

# THE DENOMINATORS OF NORMALIZED $R$ -MATRICES OF TYPES

$A_{2n-1}^{(2)}, A_{2n}^{(2)}, B_n^{(1)}$  AND  $D_{n+1}^{(2)}$

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**ABSTRACT.** Denominators of normalized  $R$ -matrices provide important information on finite dimensional integrable representations over quantum affine algebras, and over quiver Hecke algebras by the generalized quantum affine Schur-Weyl duality functors. We compute the denominators of all normalized  $R$ -matrices between fundamental representations of types  $A_{2n-1}^{(2)}, A_{2n}^{(2)}, B_n^{(1)}$  and  $D_{n+1}^{(2)}$ . Thus we can conclude that the normalized  $R$ -matrices of types  $A_{2n-1}^{(2)}, A_{2n}^{(2)}, B_n^{(1)}$  have only simple poles, and of type  $D_{n+1}^{(2)}$  have double poles under certain conditions.

## INTRODUCTION

Let  $\mathfrak{g}$  be an affine Kac-Moody algebra and  $U'_q(\mathfrak{g})$  be the quantum affine algebra corresponding to  $\mathfrak{g}$ . The finite dimensional integrable representations over  $U'_q(\mathfrak{g})$  have been investigated by many authors during the past twenty years from different perspectives (see [1, 3, 4, 10, 11, 23, 26]). Among these aspects, we focus on the theory of  $R$ -matrices which has deep relationship with  $q$ -analysis, operator algebras, conformal field theories, statistical mechanical models, etc.

The purpose of this paper is to compute the denominators of *normalized  $R$ -matrices* between the fundamental representations  $V(\varpi_{i_k})$ 's over  $U'_q(\mathfrak{g})$ . Knowing the denominators is quite crucial to study the finite dimensional integrable representations by the following theorem:

**Theorem** [1, 23] Let  $M$  be a finite dimensional irreducible integrable  $U'_q(\mathfrak{g})$ -module  $M$ . Then, there exists a finite sequence

$$((i_1, a_1), \dots, (i_l, a_l)) \text{ in } (\{1, 2, \dots, n\} \times \mathbf{k}^\times)^l$$

such that

- $d_{i_k, i_{k'}}(a_{k'}/a_k) \neq 0$  for  $1 \leq k < k' \leq l$  and
- $M$  is isomorphic to the head of  $\otimes_{i=1}^l V(\varpi_{i_k})_{a_k}$ .

Moreover, such a sequence  $((i_1, a_1), \dots, (i_l, a_l))$  is unique up to permutation. Here  $\mathbf{k} = \overline{\mathbb{C}(q)} \subset \cup_{m>0} \mathbb{C}((q^{1/m}))$  and  $d_{i_k, i_{k'}}(z) \in \mathbf{k}[z]$  denotes the denominator of the normalized  $R$ -matrix

$$R_{i_k, i_{k'}}^{\text{norm}}(z) := R_{V(\varpi_{i_k}), V(\varpi_{i_{k'}})}^{\text{norm}}(z) : V(\varpi_{i_k}) \otimes V(\varpi_{i_{k'}})_z \rightarrow \mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm 1}]} (V(\varpi_{i_{k'}})_z \otimes V(\varpi_{i_k}))$$

satisfying

$$d_{i_k, i_{k'}}(z) R_{i_k, i_{k'}}^{\text{norm}}(z) (V(\varpi_{i_k}) \otimes V(\varpi_{i_{k'}})_z) \subset V(\varpi_{i_{k'}})_z \otimes V(\varpi_{i_k}).$$

Thus the study of denominators is one of the first step to study the category  $\mathcal{C}_{\mathfrak{g}}$  consisting of finite dimensional integrable representations over  $U'_q(\mathfrak{g})$ .

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On the other hand, Kang, Kashiwara and Kim [18, 19] recently constructed *the quantum affine Schur-Weyl duality functor*  $\mathcal{F}$  by observing zeros of denominators of normalized  $R$ -matrices. The way of constructing  $\mathcal{F}$  can be described as follows: Let  $\{V_s\}_{s \in \mathcal{S}}$  be a family of fundamental representations over  $U'_q(\mathfrak{g})$ . For an index set  $J$  and two maps  $X : J \rightarrow \mathbf{k}^\times$ ,  $s : J \rightarrow \mathcal{S}$ , we can define a quiver  $Q^J = (Q_0^J, Q_1^J)$  associated with  $(J, X, s)$  as (i:vertices)  $Q_0^J = J$ , (ii:arrows) for  $i, j \in J$ , we put  $\mathbf{d}_{ij}$  many arrows from  $i$  to  $j$ , where  $\mathbf{d}_{ij}$  is the order of the zero of  $d_{V_{s(i)}, V_{s(j)}}(z)$  at  $X(j)/X(i)$ .

Then we obtain a symmetric Cartan matrix  $A^J = (a_{ij}^J)_{i,j \in J}$  associated with  $(J, X, s)$  by

$$a_{ij}^J = 2 \quad \text{if } i = j \quad \text{and} \quad a_{ij}^J = -\mathbf{d}_{ij} - \mathbf{d}_{ji} \quad \text{if } i \neq j.$$

Let  $R^J$  be the quiver Hecke algebras associated with the symmetric Cartan matrix  $A^J$  and the parameters ([24, 25, 29])

$$\mathcal{Q}_{i,j}(u, v) = (u - v)^{\mathbf{d}_{ij}}(v - u)^{\mathbf{d}_{ji}} \quad \text{if } i \neq j \quad \text{and} \quad \mathcal{Q}_{i,i}(u, v) = 0 \quad \text{for all } i \in J.$$

**Theorem** [18] There exists a functor

$$\mathcal{F} : \text{Rep}(R^J) \rightarrow \mathcal{C}_{\mathfrak{g}}$$

where  $\text{Rep}(R^J)$  denotes the category of finite dimensional representations over  $R^J$ . Moreover, the functor enjoys the following properties:

- (a)  $\mathcal{F}$  is a tensor functor; that is, there exist  $U'_q(\mathfrak{g})$ -module isomorphisms

$$\mathcal{F}(R^J(0)) \simeq \mathbf{k} \quad \text{and} \quad \mathcal{F}(M_1 \circ M_2) \simeq \mathcal{F}(M_1) \otimes \mathcal{F}(M_2)$$

for any  $M_1, M_2 \in \text{Rep}(R^J)$ .

- (b) If the Cartan matrix  $A^J$  is of type  $A_n (n \geq 1)$ ,  $D_n (n \geq 4)$ ,  $E_6$ ,  $E_7$  or  $E_8$ , then the functor  $\mathcal{F}$  is exact.

Thus the generalized quantum affine Schur-Weyl duality functor provides the way of investigating the category  $\mathcal{C}_{\mathfrak{g}}$  via the category  $\text{Rep}(R^J)$  and the other way around (see [20]).

Note that  $A^J$  depends on the choice of  $(J, X, s)$  and the denominators. Hence one may expect various exact functors defined on  $\text{Rep}(R^J)$  for a fixed algebra  $R^J$ . In the forthcoming papers by the author and his collaborators ([21, 22]), they will consider such situations, and the denominator formulas given in this paper will play an important role.

The denominators of all normalized  $R$ -matrices  $R_{k,l}^{\text{norm}}(z)$  for  $A_n^{(1)}$ ,  $C_n^{(1)}$  and  $D_n^{(1)}$  were studied in [1, 6, 19] and the denominators of normalized  $R$ -matrix  $R_{1,1}^{\text{norm}}(z)$  (resp.  $R_{n,n}^{\text{norm}}(z)$ ) between vector representations (resp. spin representations) for all classical affine types are given in [17, 27]. On the other hand, the explicit forms of normalized  $R$ -matrix  $R_{1,1}^{\text{norm}}(z)$  for all classical affine types were studied in [7, 13, 14, 15]. With these results, we will compute the denominators  $d_{k,l}(z)$  of all normalized  $R$ -matrices  $R_{k,l}^{\text{norm}}(z)$  by employing the framework given in [19, Appendix A].

Our main results are

$$d_{k,l}(z) = \prod_{s=1}^{\min(k,l)} (z^t - (-q^t)^{|k-l|+2s})(z^t - (p^*)^t(-q^t)^{2s-k-l})$$

if  $V(\varpi_k)$  and  $V(\varpi_l)$  are not spin representations, and

$$\begin{aligned} d_{k,n}(z) &= \prod_{s=1}^k (z - (-1)^{n+k} q_s^{2n-2k-1+4s}) \quad \text{if } \mathfrak{g} = B_n^{(1)} \text{ and } k < n, \\ d_{k,n}(z) &= \prod_{s=1}^k (z^2 + (-q^2)^{n-k+2s}) \quad \text{if } \mathfrak{g} = D_{n+1}^{(2)} \text{ and } k < n. \end{aligned}$$

Here,

$$t = \begin{cases} 2 & \text{if } \mathfrak{g} = D_{n+1}^{(2)}, \quad q_s^2 = q \quad \text{and } p^* := (-1)^{\langle \rho^\vee, \delta \rangle} q^{(\rho, \delta)} \\ 1 & \text{otherwise,} \end{cases}$$

for the *null root*  $\delta$  (see (1.2)). Hence we can conclude that

- (a)  $R_{k,l}^{\text{norm}}(z)$  of  $A_{2n-1}^{(2)}$ ,  $A_{2n}^{(2)}$  or  $B_n^{(1)}$  has only simple poles,
- (b)  $R_{k,l}^{\text{norm}}(z)$  of  $D_{n+1}^{(2)}$  has a double pole at  $z = (-q^2)^{s/2}$  if

$$2 \leq k, l \leq n-1, \quad k+l > n, \quad 2n+2-k-l \leq s \leq k+l \text{ and } s \equiv k+l \pmod{2},$$

- (c)  $R_{k,l}^{\text{norm}}(z)$  has a pole at  $\pm(-q^t)^{\ell/t}$  only if  $k \in \mathbb{Z}$  such that  $2 \leq \ell \leq (\rho, \delta)$  (see [8]).

This paper is organized as follows. In the first section, we recall the notion of quantum affine algebras and  $R$ -matrices, briefly. In the next section, we give the  $U'_q(\mathfrak{g})$ -module structure of the vector representations and spin representations over  $U'_q(\mathfrak{g})$ . In the third section, we study morphisms in  $\text{Hom}_{U'_q(\mathfrak{g})}(V(\varpi_i)_a \otimes V(\varpi_j)_b, V(\varpi_k)_c)$ , called the *Dorey's type morphisms*. After that, we prove the existence of certain surjective homomorphisms which can be understood as  $D_{n+1}^{(2)}$ -analogue of [19, Lemma A.3.2]. In the last section, we propose the general frame work for computing the denominators, which is originated from [19, Appendix A]. Then we compute  $d_{1,n}(z)$  for  $\mathfrak{g} = D_{n+1}^{(2)}$  and the unknown denominators  $d_{k,l}(z)$  of normalized  $R$ -matrices for  $\mathfrak{g} = A_{2n-1}^{(2)}$ ,  $A_{2n}^{(2)}$ ,  $B_n^{(1)}$  and  $D_{n+1}^{(2)}$ , by using the results in the previous sections. In the appendix, we provide a table of  $d_{k,l}(z)$  for all classical affine types for reader's convenience.

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## 1. QUANTUM AFFINE ALGEBRAS AND $R$ -MATRICES

In this section, we briefly recall the backgrounds and theories on quantum affine algebras, their finite dimensional integral representations and  $R$ -matrices. We refer to [1, 18, 23] for precise statements and definitions.

**1.1. Quantum affine algebras and their representations.** Let  $I = \{0, 1, \dots, n\}$  be a set of indices and set  $I_0 := I \setminus \{0\}$ . An *affine Cartan datum* is a quadruple  $(\mathbf{A}, \mathbf{P}, \Pi, \Pi^\vee)$  consisting of

- (a) a matrix  $\mathbf{A}$  of corank 1, called the *affine Cartan matrix* satisfying

$$(i) \ a_{ii} = 2 \ (i \in I), \quad (ii) \ a_{ij} \in \mathbb{Z}_{\leq 0}, \quad (iii) \ a_{ij} = 0 \text{ if } a_{ji} = 0$$

with  $\mathbf{D} = \text{diag}(\mathbf{d}_i \in \mathbb{Z}_{>0} \mid i \in I)$  making  $\mathbf{D}\mathbf{A}$  symmetric,

- (b) a free abelian group  $\mathbf{P}$  of rank  $n+2$ , called the *weight lattice*,
- (c)  $\Pi = \{\alpha_i \mid i \in I\} \subset \mathbf{P}$ , called the set of *simple roots*,
- (d)  $\Pi^\vee = \{h_i \mid i \in I\} \subset \mathbf{P}^\vee := \text{Hom}(\mathbf{P}, \mathbb{Z})$ , called the set of *simple coroots*,

which satisfy

- (1)  $\langle h_i, \alpha_j \rangle = a_{ij}$  for all  $i, j \in I$ ,
- (2)  $\Pi$  and  $\Pi^\vee$  are linearly independent sets,
- (3) for each  $i \in I$ , there exists  $\Lambda_i \in \mathbf{P}$  such that  $\langle h_i, \Lambda_j \rangle = \delta_{ij}$  for all  $j \in I$ .

We set  $\mathbf{Q} = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ ,  $\mathbf{Q}_+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$ ,  $\mathbf{Q}^\vee = \bigoplus_{i \in I} \mathbb{Z}h_i$  and  $\mathbf{Q}_+^\vee = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}h_i$ . We choose the *imaginary root*  $\delta = \sum_{i \in I} \mathbf{a}_i \alpha_i \in \mathbf{Q}_+$  and the *center*  $c = \sum_{i \in I} \mathbf{c}_i h_i \in \mathbf{Q}_+^\vee$  such that ([16, Chapter 4])

$$\{\lambda \in \mathbf{Q} \mid \langle h_i, \lambda \rangle = 0 \text{ for every } i \in I\} = \mathbb{Z}\delta \text{ and } \{h \in \mathbf{Q}^\vee \mid \langle h, \alpha_i \rangle = 0 \text{ for every } i \in I\} = \mathbb{Z}c.$$

Set  $\mathfrak{h} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbf{P}^\vee$ . Then there exists a symmetric bilinear form  $(\ , \ )$  on  $\mathfrak{h}^*$  satisfying

$$\langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \quad \text{for any } i \in I \text{ and } \lambda \in \mathfrak{h}^*.$$

We normalize the bilinear form by

$$\langle c, \lambda \rangle = (\delta, \lambda) \quad \text{for any } \lambda \in \mathfrak{h}^*.$$

Let us denote by  $\mathfrak{g}$  the affine Kac-Moody Lie algebra associated with  $(\mathbf{A}, \mathbf{P}, \Pi, \Pi^\vee)$  and by  $W$  the Weyl group of  $\mathfrak{g}$ , generated by  $(s_i)_{i \in I}$ . We define  $\mathfrak{g}_0$  the subalgebra of  $\mathfrak{g}$  generated by the Chevalley generators  $e_i, f_i$ , and  $h_i$  for  $i \in I_0$ . Then  $\mathfrak{g}_0$  is a finite dimensional simple Lie algebra.

Let  $\gamma$  be the smallest positive integer such that

$$\gamma(\alpha_i, \alpha_i)/2 \in \mathbb{Z} \quad \text{for any } i \in I.$$

Let  $q$  be an indeterminate. For  $m, n \in \mathbb{Z}_{\geq 0}$  and  $i \in I$ , we define  $q_i = q^{(\alpha_i, \alpha_i)/2}$  and

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^n [k]_i, \quad \begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[m-n]_i! [n]_i!}.$$

**Definition 1.1.** The *quantum affine algebra*  $U_q(\mathfrak{g})$  associated with  $(\mathbf{A}, \mathbf{P}, \Pi, \Pi^\vee)$  is the associative algebra over  $\mathbb{Q}(q^{1/\gamma})$  with 1 generated by  $e_i, f_i$  ( $i \in I$ ) and  $q^h$  ( $h \in \gamma^{-1}\mathbf{P}^\vee$ ) satisfying following relations:

- (1)  $q^0 = 1, q^h q^{h'} = q^{h+h'}$  for  $h, h' \in \gamma^{-1}\mathbf{P}^\vee$ ,
- (2)  $q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i, q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i$  for  $h \in \gamma^{-1}\mathbf{P}^\vee, i \in I$ ,
- (3)  $e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$  where  $K_i = q_i^{h_i}$ ,
- (4)  $\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(1-a_{ij}-k)} e_j e_i^{(k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(1-a_{ij}-k)} f_j f_i^{(k)} = 0$  for  $i \neq j$ ,

where  $e_i^{(k)} = e_i^k / [k]_i!$  and  $f_i^{(k)} = f_i^k / [k]_i!$ .

We denote by  $U'_q(\mathfrak{g})$  the subalgebra of  $U_q(\mathfrak{g})$  generated by  $e_i, f_i, K_i^{\pm 1}$  ( $i \in I$ ) and we call it *also* the quantum affine algebra. Throughout this paper, we mainly deal with  $U'_q(\mathfrak{g})$ .

For  $U'_q(\mathfrak{g})$ -modules  $M$  and  $N$ ,  $M \otimes N$  becomes a  $U'_q(\mathfrak{g})$ -module by the coproduct  $\Delta$  of  $U'_q(\mathfrak{g})$ :

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i.$$

Set  $\mathbf{P}_{\text{cl}} := \mathbf{P}/\mathbb{Z}\delta$  and  $\text{cl}: \mathbf{P} \rightarrow \mathbf{P}_{\text{cl}}$  as the canonical projection.

We say that a  $U'_q(\mathfrak{g})$ -module  $M$  is *integrable* provided that

- (a)  $M$  decomposes into  $\mathbf{P}_{\text{cl}}$ -weight spaces; that is,

$$M = \bigoplus_{\mu \in \mathbf{P}_{\text{cl}}} M_\mu,$$

where  $M_\mu := \{v \in M \mid K_i v = q^{\langle h_i, \mu \rangle} v\}$ ,

- (b)  $e_i$  and  $f_i$  ( $i \in I$ ) act on  $M$  nilpotently.

For  $i \in I_0$ , the level 0 fundamental weight  $\varpi_i$  is defined by

$$\varpi_i := \gcd(c_0, c_i)^{-1}(c_0\Lambda_i - c_i\Lambda_0) \in P.$$

Then  $\{\text{cl}(\varpi_i) \mid i \in I_0\}$  forms a basis for the space of *classical integral weight level 0*, denoted by  $P_{\text{cl}}^0$ , which is defined as follows:

$$P_{\text{cl}}^0 = \{\lambda \in P_{\text{cl}} \mid \langle c, \lambda \rangle = 0\}.$$

The Weyl group  $W_0$  of  $\mathfrak{g}_0$ , generated by  $(s_i)_{i \in I_0}$ , acts on  $P_{\text{cl}}^0$  (see [1, §1.2]). We denote by  $w_0$  the longest element of  $W_0$ .

**Definition 1.2.** [1, §1.3] For  $i \in I_0$ , the  $i$ th fundamental module is a unique finite dimensional integrable  $U'_q(\mathfrak{g})$ -module  $V(\varpi_i)$  satisfying the following properties:

- (1) The weights of  $V(\varpi_i)$  are contained in the convex hull of  $W_0\text{cl}(\varpi_i)$ .
- (2)  $V(\varpi_i)_{\text{cl}(\varpi_i)} = \mathbb{C}(q)\mathbf{v}_{\varpi_i}$ . (We call the vector  $\mathbf{v}_{\varpi_i}$  a *dominant integral weight vector*.)
- (3) For any  $\mu \in W_0\text{cl}(\varpi_i)$ , we can associate a non-zero vector  $u_\mu$ , called an *extremal vector of weight  $\mu$* , such that

$$(1.1) \quad \mathcal{S}_i \cdot u_\mu := u_{s_i\mu} = \begin{cases} f_i^{(\langle h_i, \mu \rangle)} u_\mu & \text{if } \langle h_i, \mu \rangle \geq 0, \\ e_i^{(-\langle h_i, \mu \rangle)} u_\mu & \text{if } \langle h_i, \mu \rangle \leq 0, \end{cases} \quad \text{for any } i \in I.$$

- (4)  $\mathbf{v}_{\varpi_i}$  generates  $V(\varpi_i)$  as a  $U'_q(\mathfrak{g})$ -module.

Let  $\mathbf{k}$  be an algebraic closure of  $\mathbb{C}(q)$  in  $\cup_{m>0}\mathbb{C}((q^{1/m}))$ . When we deal with  $U'_q(\mathfrak{g})$ -modules, we regard the base field as  $\mathbf{k}$ .

For a  $U'_q(\mathfrak{g})$ -module  $M$ , we denote by  ${}^*M$  the *right dual* and  $M^*$  the *left dual* of  $M$ , if there exist  $U'_q(\mathfrak{g})$ -homomorphisms

$$M^* \otimes M \xrightarrow{\text{tr}} \mathbf{k}, \quad \mathbf{k} \longrightarrow M \otimes M^* \quad \text{and} \quad M \otimes {}^*M \xrightarrow{\text{tr}} \mathbf{k}, \quad \mathbf{k} \longrightarrow M^* \otimes M.$$

Recall that  $V(\varpi_i)$  is finite dimensional and has the right dual and left dual as follows:

$$(1.2) \quad V(\varpi_i)^* := V(\varpi_{i^*})_{(p^*)^{-1}}, \quad {}^*V(\varpi_i) := V(\varpi_{i^*})_{p^*} \quad \text{and} \quad p^* := (-1)^{\langle \rho^\vee, \delta \rangle} q^{(\rho, \delta)}$$

where

- $*$  is the involution of  $I_0$  defined by the image of  $\varpi_i$  under the action  $w_0$ ; i.e.,  $w_0(\varpi_i) = -\varpi_{i^*}$ ,
- $\rho$  is defined by  $\langle h_i, \rho \rangle = 1$  and  $\rho^\vee$  is defined by  $\langle \rho^\vee, \alpha_i \rangle = 1$  for all  $i \in I$ .

We call an integrable  $U'_q(\mathfrak{g})$ -module  $M$  *good* if  $M$  satisfies certain properties. In this paper, the whole definition of the good module is not needed. Thus we refer [23] for the precise definition of good module. However, we would like to emphasize one condition of the good module:

A good module  $M$  contains the unique (up to constant) weight vector  $v_M$  of weight  $\lambda$ , such that

$$\text{wt}(M) \subset \lambda + \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \text{cl}(\alpha_i).$$

We call  $v_M$  the *dominant extremal weight vector* and  $\lambda$  *dominant extremal weight*. For instance, the  $i$ th fundamental representation is a good module.

For an indeterminate  $z$  and a  $U'_q(\mathfrak{g})$ -module  $M$ , let us denote by  $M_z = \{u_z \mid u \in M\}$  the  $U'_q(\mathfrak{g})$ -module  $\mathbf{k}[z^{\pm 1}] \otimes M$  with the action of  $U'_q(\mathfrak{g})$  given by

$$e_i(u_z) = z^{\delta_{i,0}}(e_i u)_z, \quad f_i(u_z) = z^{-\delta_{i,0}}(f_i u)_z, \quad K_i(u_z) = (K_i u)_z.$$

We sometimes use the notation  $u$  for  $u_z$  to simplify equations. (For example, see the proof of Proposition 4.7)

**1.2. Normalized and universal  $R$ -matrices.** We call a  $\mathbf{k}[z^{\pm 1}] \otimes U'_q(\mathfrak{g})$ -module homomorphism between  $M \otimes N_z$  and  $N_z \otimes M$  as an *intertwiner*. It is known that, for finite dimensional integral  $U'_q(\mathfrak{g})$ -modules  $M$  and  $N$ , there exists an intertwiner

$$R_{M,N}^{\text{univ}}(z) : M \otimes N_z \rightarrow N_z \otimes M$$

which satisfies

$$(1.3) \quad R_{M,N \otimes N'}^{\text{univ}}(z) = (N_z \otimes R_{M,N'}^{\text{univ}}(z)) \circ (R_{M,N}^{\text{univ}}(z) \otimes N'_z).$$

We call  $R_{M,N}^{\text{univ}}$  the *universal  $R$ -matrix* [9].

**Definition 1.3.** For good modules  $M$  and  $N$ , the *normalized  $R$ -matrix*  $R_{M,N}^{\text{norm}}$  is the  $U'_q(\mathfrak{g})$ -module homomorphism

$$(1.4) \quad R_{M,N}^{\text{norm}} : M_{z_M} \otimes N_{z_N} \rightarrow \mathbf{k}(z_M, z_N) \otimes_{\mathbf{k}[z_M^{\pm 1}, z_N^{\pm 1}]} N_{z_N} \otimes M_{z_M}$$

which satisfies

$$R_{M,N}^{\text{norm}} \circ z_M = z_M \circ R_{M,N}^{\text{norm}}, \quad R_{M,N}^{\text{norm}} \circ z_N = z_N \circ R_{M,N}^{\text{norm}} \quad \text{and} \quad R_{M,N}^{\text{norm}}(\mathbf{v}_M \otimes \mathbf{v}_N) = \mathbf{v}_N \otimes \mathbf{v}_M.$$

[1, Corollary 2.5] tells that, for good modules  $M$  and  $N$

$$\text{Hom}_{\mathbf{k}(z) \otimes U'_q(\mathfrak{g})}(M \otimes N_z, N_z \otimes M) \simeq \mathbf{k}(z),$$

and hence there exists  $a_{M,N}(z) \in \mathbf{k}(z)$  such that

$$(1.5) \quad R_{M,N}^{\text{univ}}(z) = a_{M,N}(z) R_{M,N}^{\text{norm}}(z).$$

Note that

$$R_{M,N}^{\text{norm}}(z)(M \otimes N_z) \subset \mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm 1}]} (N_z \otimes M)$$

and there exists a unique monic polynomial  $d_{M,N}(z) \in \mathbf{k}[z]$  such that

$$(1.6) \quad d_{M,N}(z) R_{M,N}^{\text{norm}}(z)(M \otimes N_z) \subset (N_z \otimes M).$$

We call  $d_{M,N}(u)$  the *denominator* of  $R_{M,N}^{\text{norm}}(z)$ .

**Lemma 1.4** ([1, Lemma C.15]). *Let  $V', V'', V$  and  $W$  be irreducible  $U'_q(\mathfrak{g})$ -modules. Assume that we have a surjective  $U'_q(\mathfrak{g})$ -homomorphism*

$$V' \otimes V'' \twoheadrightarrow V.$$

*Then we have*

$$\frac{d_{W,V'}(z)d_{W,V''}(z)a_{W,V}(z)}{d_{W,V}(z)a_{W,V'}(z)a_{W,V''}(z)} \quad \text{and} \quad \frac{d_{V',W}(z)d_{V'',W}(z)a_{V,W}(z)}{d_{V,W}(z)a_{V',W}(z)a_{V'',W}(z)} \in \mathbf{k}[z^{\pm 1}].$$

## 2. VECTOR AND SPIN REPRESENTATION.

In this section, we record the  $U'_q(\mathfrak{g})$ -module structure of

- $V(\varpi_1)$ , called the *vector representation*,
- $V(\varpi_n)$ , called the *spin representation*, for  $\mathfrak{g} = B_n^{(1)}$  or  $D_{n+1}^{(2)}$ .

As a vector space, the vector representation can be expressed as follows ([12, Chapter 11]):

$$V(\varpi_1) = \left( \bigoplus_{j=1}^n \mathbf{k}v_j \right) \oplus \left( \bigoplus_{j=1}^n \mathbf{k}v_{\bar{j}} \right) \oplus W$$

where

$\mathfrak{g}$	$A_{2n-1}^{(2)}$	$B_n^{(1)}$	$A_{2n}^{(2)}$	$D_{n+1}^{(2)}$
$W$	$\emptyset$	$\mathbf{k}v_0$	$\mathbf{k}v_{\emptyset}$	$\mathbf{k}v_0 \oplus \mathbf{k}v_{\emptyset}$

The actions of  $e_i$ ,  $f_i$  and  $q^h$  are defined by follows:

$$q^h \cdot v_j = q^{\langle h, \text{wt}(v_j) \rangle} v_j \quad \text{for } h \in P_{\text{cl}}^\vee,$$

$\mathfrak{g}$	$e_i$	$f_i$
$A_{2n-1}^{(2)}$	$e_i v_j = \begin{cases} v_i & \text{if } j = i+1 \text{ and } i \neq n, \\ v_{\overline{i+1}} & \text{if } j = \overline{i} \text{ and } i \neq n, \\ v_n & \text{if } j = \overline{n} \text{ and } i = n, \\ v_{\overline{2}} & \text{if } j = 1 \text{ and } i = 0, \\ v_{\overline{1}} & \text{if } j = 2 \text{ and } i = 0, \\ 0 & \text{otherwise,} \end{cases}$	$f_i v_j = \begin{cases} v_{i+1} & \text{if } j = i \text{ and } i \neq n, \\ v_{\overline{i}} & \text{if } j = \overline{i+1} \text{ and } i \neq n, \\ v_{\overline{n}} & \text{if } j = n \text{ and } i = n, \\ v_1 & \text{if } j = \overline{2} \text{ and } i = 0, \\ v_2 & \text{if } j = \overline{1} \text{ and } i = 0, \\ 0 & \text{otherwise,} \end{cases}$
$A_{2n}^{(2)}$	$e_i v_j = \begin{cases} v_i & \text{if } j = i+1 \text{ and } i \neq n, \\ v_{\overline{i+1}} & \text{if } j = \overline{i} \text{ and } i \neq n, \\ v_n & \text{if } j = \overline{n} \text{ and } i = n, \\ v_\emptyset & \text{if } j = 1 \text{ and } i = 0, \\ [2]_0 v_{\overline{1}} & \text{if } j = \emptyset \text{ and } i = 0, \\ 0 & \text{otherwise,} \end{cases}$	$f_i v_j = \begin{cases} v_{i+1} & \text{if } j = i \text{ and } i \neq n, \\ v_{\overline{i}} & \text{if } j = \overline{i+1} \text{ and } i \neq n, \\ v_{\overline{n}} & \text{if } j = n \text{ and } i = n, \\ v_\emptyset & \text{if } j = \overline{1} \text{ and } i = 0, \\ [2]_0 v_1 & \text{if } j = \emptyset \text{ and } i = 0, \\ 0 & \text{otherwise,} \end{cases}$
$B_n^{(1)}$	$e_i v_j = \begin{cases} v_i & \text{if } j = i+1 \text{ and } i \neq n, \\ v_{\overline{i+1}} & \text{if } j = \overline{i} \text{ and } i \neq n, \\ v_0 & \text{if } j = \overline{n} \text{ and } i = n, \\ [2]_n v_n & \text{if } j = 0 \text{ and } i = n, \\ v_{\overline{2}} & \text{if } j = 1 \text{ and } i = 0, \\ v_{\overline{1}} & \text{if } j = 2 \text{ and } i = 0, \\ 0 & \text{otherwise,} \end{cases}$	$f_i v_j = \begin{cases} v_{i+1} & \text{if } j = i \text{ and } i \neq n, \\ v_{\overline{i}} & \text{if } j = \overline{i+1} \text{ and } i \neq n, \\ v_0 & \text{if } j = n \text{ and } i = n, \\ [2]_n v_{\overline{n}} & \text{if } j = 0 \text{ and } i = n, \\ v_1 & \text{if } j = \overline{2} \text{ and } i = 0, \\ v_2 & \text{if } j = \overline{1} \text{ and } i = 0, \\ 0 & \text{otherwise,} \end{cases}$
$D_{n+1}^{(2)}$	$e_i v_j = \begin{cases} v_i & \text{if } j = i+1 \text{ and } i \neq n, \\ v_{\overline{i+1}} & \text{if } j = \overline{i} \text{ and } i \neq n, \\ v_0 & \text{if } j = \overline{n} \text{ and } i = n, \\ [2]_n v_n & \text{if } j = 0 \text{ and } i = n, \\ v_\emptyset & \text{if } j = 1 \text{ and } i = 0, \\ [2]_0 v_{\overline{1}} & \text{if } j = \emptyset \text{ and } i = 0, \\ 0 & \text{otherwise,} \end{cases}$	$f_i v_j = \begin{cases} v_{i+1} & \text{if } j = i \text{ and } i \neq n, \\ v_{\overline{i}} & \text{if } j = \overline{i+1} \text{ and } i \neq n, \\ v_0 & \text{if } j = n \text{ and } i = n, \\ [2]_n v_{\overline{n}} & \text{if } j = 0 \text{ and } i = n, \\ v_\emptyset & \text{if } j = \overline{1} \text{ and } i = 0, \\ [2]_0 v_1 & \text{if } j = \emptyset \text{ and } i = 0, \\ 0 & \text{otherwise,} \end{cases}$

where

$$\text{wt}(v_j) = \epsilon_j, \text{wt}(v_{\overline{j}}) = -\epsilon_j \text{ for } j = 1, \dots, n \text{ and } \text{wt}(v_0) = \text{wt}(v_\emptyset) = 0.$$

For  $\mathfrak{g} = B_n^{(1)}$  or  $\mathfrak{g} = D_{n+1}^{(2)}$ , the spin representation  $V(\varpi_n)$  is the  $\mathbf{k}$ -vector space with a basis

$$\mathbf{B}_{\text{sp}} = \{(m_1, \dots, m_n); m_i = + \text{ or } -\}.$$

Its  $U'_q(\mathfrak{g})$ -module structure is given by defining the action of  $e_i$ ,  $f_i$  and  $q^h$  as follows:

$$q^h v = q^{\langle h, \text{wt}(v) \rangle} v \quad \text{for } h \in P_{\text{cl}}^\vee, \text{ where } \text{wt}(v) = \frac{1}{2} \sum_{k=1}^n m_k \epsilon_k,$$

$\mathfrak{g}$	$e_i, f_i$
$B_n^{(1)}$	$e_i v = \begin{cases} (m_1, \dots, \overset{i}{+}, \overset{i+1}{-}, \dots, m_n) & \text{if } i \neq n \text{ and } m_i = -, m_{i+1} = +, \\ (m_1, \dots, m_{n-1}, \overset{n}{+}) & \text{if } i = n \text{ and } m_n = -, \\ (-, -, m_3, \dots, m_n) & \text{if } i = 0 \text{ and } m_1 = m_2 = +, \\ 0 & \text{otherwise,} \end{cases}$
	$f_i v = \begin{cases} (m_1, \dots, \overset{i}{-}, \overset{i+1}{+}, \dots, m_n) & \text{if } i \neq n \text{ and } m_i = +, m_{i+1} = -, \\ (m_1, \dots, m_{n-1}, \overset{n}{-}) & \text{if } i = n \text{ and } m_n = +, \\ (+, +, m_3, \dots, m_n) & \text{if } i = 0 \text{ and } m_1 = m_2 = -, \\ 0 & \text{otherwise,} \end{cases}$
$D_{n+1}^{(2)}$	$e_i v = \begin{cases} (m_1, \dots, \overset{i}{+}, \overset{i+1}{-}, \dots, m_n) & \text{if } i \neq n \text{ and } m_i = -, m_{i+1} = +, \\ (m_1, \dots, m_{n-1}, \overset{n}{+}) & \text{if } i = n \text{ and } m_n = -, \\ (-, m_2, \dots, m_n) & \text{if } i = 0 \text{ and } m_1 = +, \\ 0 & \text{otherwise,} \end{cases}$
	$f_i v = \begin{cases} (m_1, \dots, \overset{i}{-}, \overset{i+1}{+}, \dots, m_n) & \text{if } i \neq n \text{ and } m_i = +, m_{i+1} = -, \\ (m_1, \dots, m_{n-1}, \overset{n}{-}) & \text{if } i = n \text{ and } m_n = +, \\ (+, m_2, \dots, m_n) & \text{if } i = 0 \text{ and } m_1 = -, \\ 0 & \text{otherwise.} \end{cases}$

### 3. SURJECTIVE HOMOMORPHISMS BETWEEN INTEGRABLE $U'_q(\mathfrak{g})$ -MODULES

In this section, we first study the morphisms in

$$\text{Hom}_{U'_q(\mathfrak{g})}(V(\varpi_i)_a \otimes V(\varpi_j)_b, V(\varpi_k)_c) \quad \text{for } i, j, k \in I_0 \text{ and } a, b, c \in \mathbf{k}^\times.$$

These kinds of morphisms are known as *Dorey's type morphisms* and have been investigated in [5] for the classical untwisted affine types  $A_n^{(1)}$ ,  $B_n^{(1)}$ ,  $C_n^{(1)}$  and  $D_n^{(1)}$ . In the last part of this section, we study the surjective homomorphisms which can be understood as  $D_{n+1}^{(2)}$ -analogue of the surjective homomorphisms given in [19, (A.17)]

Hereafter, we will use the following convention frequently:

For a statement  $P$ ,  $\delta(P)$  is 1 if  $P$  is true and 0 if  $P$  is false.

By the result on  $B_n^{(1)}$  in [5], it suffices to consider when  $\mathfrak{g} = A_{2n-1}^{(2)}$  ( $n \geq 3$ ),  $A_{2n}^{(2)}$  ( $n \geq 2$ ) and  $D_{n+1}^{(2)}$  ( $n \geq 2$ ).

The finite Dynkin diagrams of  $\mathfrak{g}_0$  associated with  $\mathfrak{g}$  are given as follows:

$$C_n: \begin{array}{c} \epsilon_1 - \epsilon_2 \\ \circ_1 \end{array} \cdots \begin{array}{c} \epsilon_{n-1} - \epsilon_n \\ \circ_{n-1} \end{array} \xrightarrow{\epsilon_n} \begin{array}{c} \epsilon_n \\ \circ_n \end{array} \quad (A_{2n-1}^{(2)}, A_{2n}^{(2)}) \quad B_n: \begin{array}{c} \epsilon_1 - \epsilon_2 \\ \circ_1 \end{array} \cdots \begin{array}{c} \epsilon_{n-1} - \epsilon_n \\ \circ_{n-1} \end{array} \xleftarrow{2\epsilon_n} \begin{array}{c} 2\epsilon_n \\ \circ_n \end{array} \quad (D_{n+1}^{(2)}).$$

We denote by  $V_0(\varpi_i)$  for  $i \in I_0$ , the highest weight  $U_q(\mathfrak{g}_0)$ -module with the highest weight  $\varpi_i$ .

Throughout this paper, we set

$$(3.1) \quad t = \begin{cases} 2 & \text{if } \mathfrak{g} = D_{n+1}^{(2)}, \\ 1 & \text{otherwise,} \end{cases} \quad \text{and} \quad \vartheta = \begin{cases} 1 & \text{if } \mathfrak{g} = B_n^{(1)} \text{ or } D_{n+1}^{(2)}, \\ 0 & \text{otherwise.} \end{cases}$$



3.1.  $i + j = k \leq n - \vartheta$ . Recall that there exists an injective  $U_q(\mathfrak{g}_0)$ -module homomorphism (see. [12, Chapter 8])

$$\Phi_{i,j} : V_0(\varpi_{i+j}) \hookrightarrow V_0(\varpi_i) \otimes V_0(\varpi_j) \quad \text{for } i + j \leq n - \vartheta$$

given by

$$(3.2) \quad u_\lambda \mapsto v_\lambda = \sum_{\lambda=\mu+\xi} C_{\mu,\xi}^\lambda u_\mu \otimes u_\xi \quad (C_{\mu,\xi}^\lambda \in \mathbf{k})$$

where  $\lambda \in W_0 \cdot \varpi_{i+j}$  and  $\mu$  (resp.  $\xi$ ) runs over the elements in  $W_0 \cdot \varpi_i$  (resp.  $W_0 \cdot \varpi_j$ ).

For a positive integer  $l \leq n - \vartheta$ , we sometimes write  $\lambda \in \text{wt}(V_0(\varpi_l))$  as a sequence  $(\lambda_1, \dots, \lambda_n) \in \{1, 0, -1\}^n$  such that  $\lambda = \sum_{k=1}^n \lambda_k \epsilon_k$ .

In (3.2), since  $\Phi_{i,j}$  is a  $U_q(\mathfrak{g}_0)$ -homomorphism and  $V(\varpi_{i+j})$  is generated by  $u_{\varpi_{i+j}}$ , we can observe that

$$(3.3) \quad \lambda_k \geq 0 \text{ implies that } \mu_k, \xi_k \geq 0 \text{ and } \lambda_k \leq 0 \text{ implies that } \mu_k, \xi_k \leq 0.$$

Since  $\lambda_k \in \{1, 0, -1\}$ , we can conclude that

$$(3.4) \quad \mu_k \xi_k = 0 \quad \text{for all } 1 \leq k \leq n.$$

From the observation (3.3),  $C_{\mu,\xi}^\lambda$  must be the same as  $C_{s_k \mu, s_k \xi}^{s_k \lambda}$  whenever  $\langle h_k, \lambda \rangle \neq 0$ .

**Proposition 3.1.** *Set*

$$(3.5) \quad c_{\mu,\xi}^\lambda = \#\{(a, b) \mid a < b, (\mu_a, \xi_a) = (0, 1), \mu_b \neq 0\} \\ + \#\{(a, b) \mid a < b, (\mu_a, \xi_a) = (-1, 0), \xi_b \neq 0\}.$$

Then the  $C_{\mu,\xi}^\lambda$  in (3.2) is given as follows:

$$C_{\mu,\xi}^\lambda = (-q_1)^{c_{\mu,\xi}^\lambda}.$$

*Proof.* First, we check that  $c_{s_k \mu, s_k \xi}^{s_k \lambda} = c_{\mu,\xi}^\lambda$  whenever  $\langle h_k, \lambda \rangle \neq 0$ . To do this, it suffices to consider  $(a, b) = (k, k+1)$ . Then one can easily check that

$$\begin{aligned} & \#\{(a, b) \mid a < b, (\mu_a, \xi_a) = (0, 1), \mu_b \neq 0\} \\ & + \#\{(a, b) \mid a < b, (\mu_a, \xi_a) = (-1, 0), \xi_b \neq 0\} \\ & = \#\{(a, b) \mid a < b, ((s_k \mu)_a, (s_k \xi)_a) = (0, 1), (s_k \mu)_b \neq 0\} \\ & + \#\{(a, b) \mid a < b, ((s_k \mu)_a, (s_k \xi)_a) = (-1, 0), (s_k \xi)_b \neq 0\}. \end{aligned}$$

Thus we can assume that  $\lambda = \varpi_{i+j}$ . If  $k \geq i + j$ , then  $e_k v_\lambda = 0$ , trivially. When  $1 \leq k < i + j$ ,

$$0 = e_k v_\lambda = \sum_{\substack{(\mu_k, \mu_{k+1})=(0,1) \\ (\xi_k, \xi_{k+1})=(1,0)}} C_{\mu,\xi}^\lambda q_1^{-1} v_{s_k \mu} \otimes v_\xi + \sum_{\substack{(\mu_k, \mu_{k+1})=(1,0) \\ (\xi_k, \xi_{k+1})=(0,1)}} C_{\mu,\xi}^\lambda v_\mu \otimes v_{s_k \xi}.$$

Thus, for  $(\mu_k, \mu_{k+1}) = (0, 1)$  and  $(\xi_k, \xi_{k+1}) = (1, 0)$ , we have

$$c_{\mu,\xi}^\lambda = c_{s_k \mu, s_k \xi}^\lambda + 1$$

which implies our assertion.  $\square$

Now we shall determine  $x, y \in \mathbf{k}^\times$  such that there exists an injective  $U'_q(\mathfrak{g})$ -module homomorphism

$$(3.6) \quad V(\varpi_{i+j}) \hookrightarrow V(\varpi_i)_x \otimes V(\varpi_j)_y.$$

The strategy in this subsection can be explained as follows: As a  $U_q(\mathfrak{g}_0)$ -module, we have an injection

$$V_0(\varpi_{i+j}) \hookrightarrow V(\varpi_i)_x \otimes V(\varpi_j)_y.$$

By using characterization of  $V(\varpi_{i+j})$  given in [1, § 1.3], it suffices to determine  $x$  and  $y$  satisfying

$$C_{\mathfrak{s}_0\mu, \mathfrak{s}_0\xi}^{\mathfrak{s}_0\lambda} = f(x)g(y)C_{\mu, \xi}^{\lambda}$$

where

- $\lambda$ ,  $\mu$  and  $\xi$  are extremal weights and  $\langle h_0, \lambda \rangle \neq 0$ ,
- $f(x)$  and  $g(y)$  arise from the action of  $e_0$  or  $f_0$  on  $V(\varpi_i)_x$  or  $V(\varpi_j)_y$ .

Recall the notion  $\mathcal{S}_k \cdot u_\mu = u_{\mathfrak{s}_k\mu}$  for an extremal weight  $\mu$  in (1.1).

**Proposition 3.2.** *Let  $\mathfrak{g} = A_{2n-1}^{(2)}$  ( $n \geq 3$ ). Then the  $x, y$  in (3.6) are given as follows:*

$$x = (-q)^j \quad \text{and} \quad y = (-q)^{-i}.$$

*Proof.* The Dynkin diagram of  $A_{2n-1}^{(2)}$  is given as follows:

$$\begin{array}{ccccccc} & & -\epsilon_1 - \epsilon_2 & & 0 & & \\ & & | & & | & & \\ \epsilon_1 - \epsilon_2 & \epsilon_2 - \epsilon_3 & & \dots & \epsilon_{n-1} - \epsilon_n & 2\epsilon_n & \\ \circ & \circ & & \dots & \circ & \circ & \\ 1 & 2 & & & n-1 & n & \end{array}.$$

It suffices to consider  $\lambda \in W_0 \cdot \varpi_{i+j}$  such that  $\lambda_1, \lambda_2 \geq 0$ . Thus it is enough to consider when  $\mu_1, \mu_2, \xi_1, \xi_2 \geq 0$ . Then we have

$$\mathcal{S}_0 \cdot v_\lambda = v_{\mathfrak{s}_0\lambda} = \sum C_{\mu, \xi}^{\lambda} x^{\delta(\mu_1=1)+\delta(\mu_2=1)} y^{\delta(\xi_1=1)+\delta(\xi_2=1)} v_{\mathfrak{s}_0\mu} \otimes v_{\mathfrak{s}_0\xi}.$$

Here  $\mathfrak{s}_0(\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_n) = \mathfrak{s}_0(-\epsilon_2, -\epsilon_1, \epsilon_3, \dots, \epsilon_n)$  ( $\epsilon_i \in \{-1, 0, 1\}$ ). Thus

$$C_{\mathfrak{s}_0\mu, \mathfrak{s}_0\xi}^{\mathfrak{s}_0\lambda} = x^{\delta(\mu_1=1)+\delta(\mu_2=1)} y^{\delta(\xi_1=1)+\delta(\xi_2=1)} C_{\mu, \xi}^{\lambda}.$$

On the other hand, by (3.5),

$$\begin{aligned} c_{\mathfrak{s}_0\mu, \mathfrak{s}_0\xi}^{\mathfrak{s}_0\lambda} - c_{\mu, \xi}^{\lambda} &= +\delta(\mu_1 = 1) \times \#\{b > 1 \mid \xi_b \neq 0\} + \delta(\mu_2 = 1) \times \#\{b > 2 \mid \xi_b \neq 0\} \\ &\quad - \delta(\xi_1 = 1) \times \#\{b > 1 \mid \mu_b \neq 0\} - \delta(\xi_2 = 1) \times \#\{b > 2 \mid \mu_b \neq 0\} \\ &= \delta(\mu_1 = 1) \times j + \delta(\mu_2 = 1) \times (j - \delta(\xi_2 = 1)) \\ &\quad - \delta(\xi_1 = 1) \times i - \delta(\xi_2 = 1) \times (i - \delta(\mu_2 = 1)). \end{aligned}$$

By (3.4),  $\mu_i \xi_i = 0$  ( $i = 1, 2$ ) and hence we can conclude that

$$c_{\mathfrak{s}_0\mu, \mathfrak{s}_0\xi}^{\mathfrak{s}_0\lambda} - c_{\mu, \xi}^{\lambda} = -(\delta(\xi_1 = 1) + \delta(\xi_2 = 1)) \times i + (\delta(\mu_1 = 1) + \delta(\mu_2 = 1)) \times j.$$

Thus we have

$$x = (-q)^j \quad \text{and} \quad y = (-q)^{-i}.$$

□

**Proposition 3.3.** *Let  $\mathfrak{g} = A_{2n}^{(2)}$  ( $n \geq 2$ ). Then the  $x, y$  in (3.6) are given as follows:*

$$x = (-q)^j \quad \text{and} \quad y = (-q)^{-i}.$$

*Proof.* The Dynkin diagram of  $A_{2n}^{(2)}$  is given as follows:

$$\begin{array}{ccccccc} & & -\epsilon_1 & & \epsilon_1 - \epsilon_2 & & \\ & & | & & | & & \\ & & \circ & \dots & \circ & & \\ & & 0 & & 1 & & n \\ & & & & & & n-1 \end{array}$$

It suffices to consider  $\lambda \in W_0 \cdot \varpi_{i+j}$  such that  $\langle h_0, \lambda \rangle < 0$  and hence  $\lambda_1 = 1$ . Then we have

$$\mathcal{S}_0 \cdot v_\lambda = v_{\mathfrak{s}_0\lambda} = e_0^{(2)} v_\lambda = C_{\mu, \xi}^{\lambda} x^{\delta(\mu_1=1)} y^{\delta(\xi_1=1)} v_{\mathfrak{s}_0\mu} \otimes v_{\mathfrak{s}_0\xi}.$$

Here  $\mathbf{s}_0(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) = \mathbf{s}_0(-\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ . Thus

$$C_{\mathbf{s}_0\mu, \mathbf{s}_0\xi}^{\mathbf{s}_0\lambda} = x^{\delta(\mu_1=1)} y^{\delta(\xi_1=1)} C_{\mu, \xi}^{\lambda}.$$

On the other hand, by (3.5),

$$C_{\mathbf{s}_0\mu, \mathbf{s}_0\xi}^{\mathbf{s}_0\lambda} = (-q)^{\delta(\mu_1=1) \times \#\{b>1|\xi_b \neq 0\}} (-q)^{-\delta(\xi_1=1) \times \#\{b>1|\mu_b \neq 0\}} C_{\mu, \xi}^{\lambda}.$$

Thus we can conclude that

$$x = (-q)^j \quad \text{and} \quad y = (-q)^{-i}.$$

□

**Proposition 3.4.** *Let  $\mathfrak{g} = D_{n+1}^{(2)}$ . Then the  $x, y$  in (3.6) are given as follows:*

$$x = (-q^2)^{j/2} \quad \text{and} \quad y = (-q^2)^{-i/2}.$$

*Proof.* The Dynkin diagram of  $D_{n+1}^{(2)}$  is given as follows:

$$\begin{array}{ccccccc} \overset{-\epsilon_1}{\circ} & \xleftarrow{\epsilon_1 - \epsilon_2} & \overset{\epsilon_1 - \epsilon_2}{\circ} & \cdots & \cdots & \xrightarrow{\epsilon_{n-1} - \epsilon_n} & \overset{\epsilon_n}{\circ} \\ 0 & & 1 & & & & n-1 \quad n \end{array}$$

It suffices to consider  $\lambda \in W_0 \cdot \varpi_{i+j}$  such that  $\langle h_0, \lambda \rangle < 0$ . Thus we assume that  $\lambda_1 = 1$ . Note that  $q_1 = q^2$ . Then we have

$$\mathcal{S}_0 \cdot v_\lambda = v_{\mathbf{s}_0\lambda} = e_0^{(2)} v_\lambda = \sum C_{\mu, \xi}^{\lambda} x^{2\delta(\mu_1=1)} y^{2\delta(\xi_1=1)} v_{\mathbf{s}_0\mu} \otimes v_{\mathbf{s}_0\xi}.$$

Here  $\mathbf{s}_0(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) = \mathbf{s}_0(-\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ . Thus

$$C_{\mathbf{s}_0\mu, \mathbf{s}_0\xi}^{\mathbf{s}_0\lambda} = x^{2\delta(\mu_1=1)} y^{2\delta(\xi_1=1)} C_{\mu, \xi}^{\lambda}.$$

On the other hand, by (3.5),

$$C_{\mathbf{s}_0\mu, \mathbf{s}_0\xi}^{\mathbf{s}_0\lambda} = (-q^2)^{\delta(\mu_1=1) \times \#\{b>1|\xi_b \neq 0\}} (-q^2)^{-\delta(\xi_1=1) \times \#\{b>1|\mu_b \neq 0\}} C_{\mu, \xi}^{\lambda}.$$

Thus we can conclude that

$$x^2 = (-q^2)^j \quad \text{and} \quad y^2 = (-q^2)^{-i},$$

which yield our assertion. □

**Theorem 3.5.** *For  $i + j = k \leq n - \vartheta$ , there exists a surjective  $U'_q(\mathfrak{g})$ -module homomorphism*

$$(3.7) \quad p_{i,j}: V(\varpi_i)_{(-q^t)^{-j/t}} \otimes V(\varpi_j)_{(-q^t)^{i/t}} \twoheadrightarrow V(\varpi_k).$$

*By taking dual, there exists an injective  $U'_q(\mathfrak{g})$ -module homomorphism*

$$(3.8) \quad \iota_{i,j}: V(\varpi_k) \hookrightarrow V(\varpi_i)_{(-q^t)^{j/t}} \otimes V(\varpi_j)_{(-q^t)^{-i/t}}.$$

*Proof.* The proof immediately comes from the previous propositions. □

3.2.  $i = j = n$ ,  $k < n$  for  $\mathfrak{g} = D_{n+1}^{(2)}$ . In this subsection, we fix  $\mathfrak{g}$  as  $D_{n+1}^{(2)}$ . Recall that there exists an injective  $U_q(B_n)$ -module homomorphism (see. [12, Chapter 8])

$$V_0(\varpi_i) \hookrightarrow V_0(\varpi_n) \otimes V_0(\varpi_n)$$

given by

$$(3.9) \quad u_\lambda \mapsto v_\lambda = \sum_{\lambda=\mu+\xi} C_{\mu,\xi}^\lambda u_\mu \otimes u_\xi$$

where  $\lambda \in W_0 \cdot \varpi_i$  and  $\mu, \xi \in W_0 \cdot \varpi_n$ .

We sometimes write  $\mu \in \text{wt}(V_0(\varpi_n))$  as a sequence  $(\mu_1, \dots, \mu_n) \in \{+, -\}^n$  such that

$$\mu = \sum_{k=1}^n \frac{\mu_k}{2} \epsilon_k.$$

**Proposition 3.6.** *Set*

$$(3.10) \quad \begin{aligned} 1c_{\mu,\xi}^\lambda &= \#\{(a, b) \mid a < b, (\mu_a, \xi_a) = (-, +), (\mu_b, \xi_b) = (+, -)\}, \\ 2c_{\mu,\xi}^\lambda &= \#\{a \mid (\mu_a, \xi_a) = (-, +)\}, \\ \varphi(c) &= (-q)^c (-q^2)^{\frac{c(c-1)}{2}}. \end{aligned}$$

Then  $C_{\mu,\xi}^\lambda$  in (3.9) is given as follows:

$$C_{\mu,\xi}^\lambda = (-q^2)^{1c_{\mu,\xi}^\lambda} \varphi(2c_{\mu,\xi}^\lambda).$$

*Proof.* As in Proposition 3.1, one can check that  $C_{s_k \mu, s_k \xi}^{s_k \lambda} = C_{\mu,\xi}^\lambda$  whenever  $\langle h_k, \lambda \rangle \neq 0$ . Thus we can assume that  $\lambda = \varpi_i$ . If  $k \leq i$ , then  $e_k u_\lambda = 0$ , trivially. Thus, for  $k > i$ , we have

$$0 = e_k v_\lambda = \begin{cases} \sum_{\substack{(\mu_k, \mu_{k+1})=(-, +) \\ (\xi_k, \xi_{k+1})=(+, -)}} C_{\mu,\xi}^\lambda (q^2)^{-1} u_{s_k \mu} \otimes u_\xi \\ \quad + \sum_{\substack{(\mu_k, \mu_{k+1})=(+, -) \\ (\xi_k, \xi_{k+1})=(-, +)}} C_{\mu,\xi}^\lambda u_\mu \otimes u_{s_k \xi} & \text{if } i < k < n, \\ \sum_{(\mu_n, \xi_n)=(-, +)} C_{\mu,\xi}^\lambda q^{-1} u_{s_n \mu} \otimes u_\xi \\ \quad + \sum_{(\mu_n, \xi_n)=(+, -)} u_\mu \otimes u_{s_n \xi} & \text{if } k = n. \end{cases}$$

Thus we have

$$C_{\mu,\xi}^\lambda = \begin{cases} -q^2 C_{s_k \mu, s_k \xi}^\lambda & \text{if } i < k < n \text{ and } (\mu_k, \xi_k) = (-, +), \\ (-q)^{-1} C_{s_n \mu, s_n \xi}^\lambda & \text{if } k = n \text{ and } \mu_n = +. \end{cases}$$

On the other hand, for  $i < k < n$  and  $(\mu_k, \xi_k) = (-, +)$ , we have

$$\begin{aligned} 1c_{\mu,\xi}^\lambda &= \begin{cases} 1c_{s_k \mu, s_k \xi}^\lambda - 1 & \text{if } i < k < n \text{ and } (\mu_k, \xi_k) = (-, +), \\ 1c_{\mu,\xi}^\lambda = 1c_{s_k \mu, s_k \xi}^\lambda + 2c_{\mu,\xi}^\lambda & \text{if } k = n \text{ and } \mu_n = +, \end{cases} \\ 2c_{\mu,\xi}^\lambda &= \begin{cases} 2c_{s_k \mu, s_k \xi}^\lambda & \text{if } i < k < n \text{ and } (\mu_k, \xi_k) = (-, +), \\ 2c_{\mu,\xi}^\lambda = 2c_{s_k \mu, s_k \xi}^\lambda - 1 & \text{if } k = n \text{ and } \mu_n = +, \end{cases} \end{aligned}$$

which yield our assertion.  $\square$

**Theorem 3.7.** *For  $k \leq n - 1$ , there exists a surjective  $U'_q(D_{n+1}^{(2)})$ -module homomorphism*

$$(3.11) \quad p_{n,k}: V(\varpi_n)_{\pm\sqrt{-1}(-q^2)^{-\frac{n-k}{2}}} \otimes V(\varpi_n)_{\mp\sqrt{-1}(-q^2)^{\frac{n-k}{2}}} \twoheadrightarrow V(\varpi_k).$$

By taking dual, there exists an injective  $U'_q(D_{n+1}^{(2)})$ -module homomorphism

$$(3.12) \quad \iota_{n,k}: V(\varpi_k) \hookrightarrow V(\varpi_n)_{\pm\sqrt{-1}(-q^2)^{\frac{n-k}{2}}} \otimes V(\varpi_n)_{\mp\sqrt{-1}(-q^2)^{-\frac{n-k}{2}}}.$$

*Proof.* We apply the same strategy of §3.1; i.e., we determine the  $x$  and  $y$  in (3.6). As in Proposition 3.4, we first consider  $\lambda \in W_0 \cdot \varpi_k$  with  $\lambda_1 = 1$  and hence  $\mu_1 = \xi_1 = +$ . In this case, we have

$$\mathcal{S}_0 \cdot v_\lambda = v_{s_0\lambda} = e_0^{(2)} v_\lambda = \sum C_{\mu,\xi}^\lambda xy u_{s_0\mu} \otimes u_{s_0\xi}.$$

On the other hand

$${}_1c_{\mu,\xi}^\lambda = {}_1c_{s_0\mu,s_0\xi}^{s_0\lambda}, \quad {}_2c_{\mu,\xi}^\lambda = {}_2c_{s_0\mu,s_0\xi}^{s_0\lambda}.$$

Thus we conclude that

$$xy = 1.$$

Consider  $\lambda \in W_0 \cdot \varpi_i$  with  $\langle h_0, \lambda \rangle = 0$ . Equivalently  $\lambda_1 = 0$  and hence  $-\mu_1 = \xi_1$ . In this case,

$$0 = e_0 v_\lambda = \sum_{(\mu_1, \xi_1) = (+, -)} C_{\mu,\xi}^\lambda q^{-1} x u_{s_0\mu} \otimes u_\xi + \sum_{(\mu_1, \xi_1) = (-, +)} C_{\mu,\xi}^\lambda y u_\mu \otimes u_{s_0\xi}.$$

Thus, for  $\mu_1 = +$ , we have

$$C_{s_0\mu,s_0\xi}^\lambda = C_{\mu,\xi}^\lambda (-q)^{-1} \times \frac{x}{y} = C_{\mu,\xi}^\lambda (-q)^{-1} \times x^2.$$

On the other hand,

$${}_1c_{s_0\mu,s_0\xi}^\lambda = {}_1c_{\mu,\xi}^\lambda + \#\{b > 1 \mid (\mu_b, \xi_b) = (+, -)\} \text{ and } {}_2c_{s_0\mu,s_0\xi}^\lambda = {}_2c_{\mu,\xi}^\lambda + 1.$$

Thus we have

$$x^2 = -(-q^2)^{n-k},$$

which yields our assertion.  $\square$

**3.3.  $j = 1$  and  $i = k = n$  for  $\mathfrak{g} = A_{2n}^{(2)}$ .** In this subsection, we show that there exists a surjective  $U'_q(A_{2n}^{(2)})$ -homomorphism

$$(3.13) \quad V(\varpi_n)_{(-q)^{-1}} \otimes V(\varpi_1)_{(-q)^n} \twoheadrightarrow V(\varpi_n).$$

Indeed, we do not use (3.13) in this paper. However, for the forthcoming works, we present the existence of such a homomorphism.

Similar to the previous subsections, we determine the relations among  $a$ ,  $b$  and  $c$  such that

$$(3.14) \quad V(\varpi_n)_a \hookrightarrow V(\varpi_1)_b \otimes V(\varpi_n)_c.$$

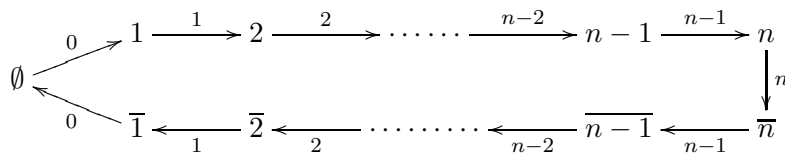
Recall that for  $k \in I_0$  (see [28, Table 1]),

$$V(\varpi_k) \simeq \bigoplus_{j=0}^k V_0(\varpi_j) \text{ as a } U_q(C_n)\text{-module.}$$

Here  $V_0(\varpi_0)$  is the trivial  $U'_q(\mathfrak{g})$ -module  $\mathbf{k}$ . Thus

$$V_0(\varpi_n)^{\oplus 2} \hookrightarrow V(\varpi_1) \otimes V(\varpi_n) \text{ as a } U_q(C_n)\text{-module.}$$

The crystal graph of  $V(\varpi_1)$  is given by (see [12, Example 11.1.4])



We denote by

$\mathbf{u}$  the dominant integral weight vector of  $V(\varpi_n)$  with its weight  $\varpi_n = \sum_{i \in I_0} \epsilon_i$ .

For  $i_1, \dots, i_k, j_1, \dots, j_l \in I_0$ , we set  $\mathbf{u}[\widehat{i_1}, \dots, \widehat{i_k}, \widehat{j_1}, \dots, \widehat{j_l}]$  the vector in  $V_0(\varpi_{n-l})$  with its weight given by

$$\text{wt}(\mathbf{u}[\widehat{i_1}, \dots, \widehat{i_k}, \widehat{j_1}, \dots, \widehat{j_l}]) = \text{wt}(\mathbf{u}) - \sum_{s=1}^k 2\epsilon_{i_s} - \sum_{t=1}^l \epsilon_{j_t},$$

if such a weight vector exists in  $V_0(\varpi_{n-l})$ .

The map (3.14), if it exists, sends  $\mathbf{u}$  to the following vector, say  $\mathbf{v}$ :

$$\mathbf{u} \mapsto \mathbf{v} = v_\emptyset \otimes \mathbf{u} + (-qb^{-1}c) \left( \sum_{k=1}^n (-q)^{k-1} v_k \otimes \mathbf{u}[\widehat{k}] \right),$$

which is unique (up to constant) in the sense that it satisfies  $e_i \mathbf{v} = 0$  for  $i \in I_0$ , and  $f_0 \mathbf{u} = 0$ .

In  $V(\varpi_n)_a$ , we have

$$(3.15) \quad \mathcal{S}_0 \cdot \mathbf{u} = a\mathcal{S}_w \cdot \mathbf{u} \quad \text{where } \mathcal{S}_w = \mathcal{S}_1 \mathcal{S}_2 \cdots \mathcal{S}_n \text{ for } w = \mathbf{s}_1 \mathbf{s}_2 \cdots \mathbf{s}_n \in W_0.$$

On the other hand,

$$\begin{aligned} \mathcal{S}_0 \cdot \mathbf{v} &= e_0^{(2)} \mathbf{v} = c v_{\overline{1}} \otimes \mathbf{u}[\widehat{1}] - qc v_\emptyset \otimes \mathbf{u}[\widehat{1}] - qb^{-1}cd \sum_{k \neq 1} (-q)^{k-1} v_k \otimes \mathbf{u}[\widehat{1}, \widehat{k}], \\ \mathcal{S}_w \cdot \mathbf{v} &= f_1^{(2)} f_2^{(2)} \cdots f_{n-1}^{(2)} f_n \mathbf{v} = v_\emptyset \otimes \mathbf{u}[\widehat{1}] + (-qb^{-1}c) \left( \sum_{k \neq n} (-q)^{k-1} v_{k+1} \otimes \mathbf{u}[\widehat{1}, \widehat{k+1}] \right) \\ &\quad + (-qb^{-1}c)(-q)^{n-1} v_{\overline{1}} \otimes \mathbf{u}[\widehat{1}], \end{aligned}$$

where  $d$  is an element in  $\mathbf{k}^\times$  such that

$$(3.16) \quad e_0^{(2)} \mathbf{u}[\widehat{k}] = d \times \mathbf{u}[\widehat{1}, \widehat{k}] \quad \text{for } k \neq 1 \text{ in } V(\varpi_n)_c.$$

By (3.15), we can conclude that

$$(3.17) \quad a = -qc, \quad b = a(-q)^n, \quad d = c.$$

Now, it suffices to show that  $d = c = 1$ .

**Proposition 3.8.** *For  $1 \neq k \in I_0$ , the coefficient  $d$  in (3.16) must be equal to 1; i.e.,*

$$e_0^{(2)} \mathbf{u}[\widehat{k}] = \mathbf{u}[\widehat{1}, \widehat{k}] \quad \text{in } V(\varpi_n)_c.$$

*Proof.* By Definition 1.1 (3), we have

$$f_1 e_0 \mathbf{u}[\widehat{2}] = e_0 f_1 \mathbf{u}[\widehat{2}] = e_0 \mathbf{u}[\widehat{1}] = [2]_0 \mathbf{u}[\widehat{1}].$$

Thus

$$e_1 e_0 \mathbf{u}[\widehat{2}] = [2]_0 e_1^{(2)} \mathbf{u}[\widehat{1}] = [2]_0 \mathbf{u}[\widehat{2}].$$

From the actions  $e_i$  ( $i \in I$ ) on  $V(\varpi_n)_c$ , we have

$$(3.18) \quad e_0 e_1 e_0^{(2)} \mathbf{u}[\widehat{2}] = c e_0 e_1 \mathbf{u}[\widehat{1}, \widehat{2}] = c e_0 \mathbf{u}[\widehat{2}, \widehat{1}] = c [2]_0 \mathbf{u}[\widehat{1}, \widehat{2}].$$

Since all vectors in  $V(\varpi_n)$  are annihilated by the action  $e_0^{(3)}$ , the relation in Definition 1.1 (4) implies that

$$(3.19) \quad e_0 e_1 e_0^{(2)} u[\widehat{2}] = (e_1 e_0^{(3)} + e_0^{(2)} e_1 e_0 - e_0^{(3)} e_1) u[\widehat{2}] = e_0^{(2)} e_1 e_0 u[\widehat{2}] = [2]_0 e_0^{(2)} u[\widehat{2}] = [2]_0 u[\overline{1}, \overline{2}].$$

From (3.17), (3.18) and (3.19), we can conclude that

$$d = c = 1.$$

□

Now, we have the following theorem.

**Theorem 3.9.** *There exists a surjective  $U'_q(A_{2n}^{(2)})$ -module homomorphism*

$$(3.20) \quad p_{1,n}: V(\varpi_n)_{(-q)^{-1}} \otimes V(\varpi_1)_{(-q)^n} \twoheadrightarrow V(\varpi_n).$$

*By taking dual, there exists an injective  $U'_q(A_{2n}^{(2)})$ -module homomorphism*

$$(3.21) \quad \iota_{1,n}: V(\varpi_n) \hookrightarrow V(\varpi_1)_{(-q)^n} \otimes V(\varpi_n)_{(-q)^{-1}}.$$

**3.4.  $D_{n+1}^{(2)}$ -analogue of the surjective homomorphisms given in [19, (A.17)].** This subsection is devoted to prove the following lemma.

**Lemma 3.10.** *Let  $\eta, \eta' \in \{\sqrt{-1}, -\sqrt{-1}\}$  and  $1 \leq k, l \leq n-1$  such that  $k+l=n$ . Then there exists a surjective  $U'_q(D_{n+1}^{(2)})$ -module homomorphism*

$$V(\varpi_k)_{\eta(-q^2)^{-\frac{1}{2}}} \otimes V(\varpi_l)_{\eta'(-q^2)^{\frac{k}{2}}} \twoheadrightarrow V(\varpi_n)_{-1} \otimes V(\varpi_n).$$

*Proof.* Note that  $\eta/\eta' = \pm 1$ . By Theorem 3.7, there are two injective  $U'_q(D_{n+1}^{(2)})$ -homomorphisms

$$\begin{aligned} \psi_1: V(\varpi_k)_{\eta(-q^2)^{-\frac{1}{2}}} &\hookrightarrow V(\varpi_n)_{-1} \otimes V(\varpi_n)_{(-q^2)^{\frac{-n+k-l}{2}}}, \\ \psi_2: V(\varpi_l)_{\eta'(-q^2)^{\frac{k}{2}}} &\hookrightarrow V(\varpi_n)_{(-q^2)^{\frac{n+k-l}{2}}} \otimes V(\varpi_n), \end{aligned}$$

by taking dual. Then we can obtain  $\varphi = (\text{id}_{V(\varpi_n)_{-1}} \otimes \text{tr} \otimes \text{id}_{V(\varpi_n)_{-1}}) \circ (\psi_1 \otimes \psi_2)$ ,

$$\varphi: V(\varpi_k)_{\eta(-q^2)^{-\frac{1}{2}}} \otimes V(\varpi_l)_{\eta'(-q^2)^{\frac{k}{2}}} \longrightarrow V(\varpi_n)_{-1} \otimes V(\varpi_n),$$

since  $V(\varpi_n)_{(-q^2)^{\frac{-n+k-l}{2}}}$  and  $V(\varpi_n)_{(-q^2)^{\frac{n+k-l}{2}}}$  are dual to each other.

Applying the argument of [19, Lemma A.3.2], we have

$$\varphi(v \otimes w) \equiv \text{tr}(u_{-\varpi_n} \otimes u_{\varpi_n}) v_1 \otimes w_1 \pmod{\bigoplus_{\lambda \neq -\varpi_k + \varpi_n} (V(\varpi_n)_{-1})_\lambda \otimes V(\varpi_n)_{-\varpi_k + \varpi_l - \lambda}},$$

where

- $v$  is the  $U_q(B_n)$ -lowest weight vector of  $V(\varpi_k)_{\eta(-q^2)^{-\frac{1}{2}}}$  of weight  $-\varpi_k$ ,
- $w$  is the  $U_q(B_n)$ -highest weight vector of  $V(\varpi_l)_{\eta'(-q^2)^{\frac{k}{2}}}$  of weight  $\varpi_l$ ,
- $v_1$  is a non-zero vector of  $V(\varpi_n)_{-1}$  of weight  $-\varpi_k + \varpi_n$ ,
- $w_1$  is a non-zero vector of  $V(\varpi_n)$  of weight  $\varpi_l - \varpi_n$ .

Thus  $\varphi$  is non-zero. Then our assertion follows from the fact that  $V(\varpi_n)_{-1} \otimes V(\varpi_n)$  is irreducible. □

#### 4. THE COMPUTATION OF DENOMINATORS BETWEEN FUNDAMENTAL REPRESENTATIONS

For simplicity, we write  $R_{k,l}^{\text{norm}}$  for  $R_{V(\varpi_k), V(\varpi_l)}^{\text{norm}}$  in (1.4),  $d_{k,l}$  for  $d_{V(\varpi_k), V(\varpi_l)}$  in (1.6) and  $a_{k,l}$  for  $a_{V(\varpi_k), V(\varpi_l)}$  in (1.5).

By the result of [1, Appendix A] and [2], the denominator  $d_{k,l}(z)$  and the element  $a_{k,l}(z) \in \mathbf{k}(z)$  are symmetric with respect to the indices  $k$  and  $l$ ; that is,

$$(4.1) \quad d_{k,l}(z) = d_{l,k}(z) \quad \text{and} \quad a_{k,l}(z) = a_{l,k}(z).$$

**4.1. General framework.** In this subsection, we propose the strategy for computing  $d_{k,l}(z)$ , which is originated from [19, Appendix A].

Note that we have a surjective homomorphism

$$(4.2) \quad p_{l-1,1}: V(\varpi_{l-1})_{(-q^t)^{-1/t}} \otimes V(\varpi_1)_{(-q^t)^{l-1/t}} \twoheadrightarrow V(\varpi_l) \quad \text{if } l \leq n - \vartheta,$$

by the previous section.

**Assumption 4.1.** Assume the followings:

- (A) We know  $a_{k,l'}(z)$  for  $k \in I_0$  and  $l' \leq l-1$ .
- (B) We know  $d_{1,1}(z)$  for all  $\mathfrak{g}$ , and  $d_{1,n}(z)$  for  $\mathfrak{g} = B_n^{(1)}$  or  $\mathfrak{g} = D_{n+1}^{(2)}$ .

With these assumptions and (1.3), consider the following commutative diagram:

$$(4.3) \quad \begin{array}{ccc} V(\varpi_k) \otimes V(\varpi_{l-1})_{(-q^t)^{-1/t}z} \otimes V(\varpi_1)_{(-q^t)^{l-1/t}z} & \xrightarrow{V(\varpi_k) \otimes p_{l-1,1}} & V(\varpi_k) \otimes V(\varpi_l)_z \\ \downarrow R_{k,l-1}^{\text{univ}}((-q^t)^{-1/t}z) \otimes V(\varpi_1)_{(-q^t)^{l-1/t}z} & & \downarrow R_{k,l}^{\text{univ}}(z) \\ V(\varpi_{l-1})_{(-q^t)^{-1/2}z} \otimes V(\varpi_k) \otimes V(\varpi_1)_{(-q^t)^{l-1/2}z} & & \\ \downarrow V(\varpi_{l-1})_{(-q^t)^{-1/t}z} \otimes R_{k,1}^{\text{univ}}((-q^t)^{l-1/t}z) & & \\ V(\varpi_{l-1})_{(-q^t)^{-1/t}z} \otimes V(\varpi_1)_{(-q^t)^{l-1/t}z} \otimes V(\varpi_k) & \xrightarrow{p_{l-1,1} \otimes V(\varpi_k)} & V(\varpi_l)_z \otimes V(\varpi_k). \end{array}$$

Then we have

$$(4.4) \quad \begin{array}{ccc} v_{[1,\dots,k]} \otimes v_{[1,\dots,l-1]} \otimes v_l & \xrightarrow{\quad} & v_{[1,\dots,k]} \otimes v_{[1,\dots,l-1,l]} \\ \downarrow & & \downarrow \\ a_{k,l-1}((-q^t)^{-1/t}z) v_{[1,\dots,l-1]} \otimes v_{[1,\dots,k]} \otimes v_l & & \\ \downarrow & & \downarrow \\ a_{k,l-1}((-q^t)^{-1/t}z) a_{k,1}((-q^t)^{l-1/t}z) v_{[1,\dots,l-1]} \otimes w & \xrightarrow{\quad} & a_{k,l}(z) v_{[1,\dots,l-1,l]} \otimes v_{[1,\dots,k]}, \end{array}$$

where

- $v_{[1,\dots,a]}$  is the dominant extremal weight vector of  $V(\varpi_a)$  for  $a \in I_0$ ,
- $w = R_{k,1}^{\text{norm}}((-q^t)^{l-1/t}z)(v_{[1,\dots,k]} \otimes v_l)$ .

By observing the vector  $w$ , we can get an equation explaining the relationship between

$$a_{k,l-1}(-q^{-1}z) a_{k,1}((-q)^{l-1}z) \quad \text{and} \quad a_{k,l}(z).$$

By Assumption 4.1 (A), we can compute  $a_{k,l}(z)$  by using an induction.



After getting  $a_{k,l}(z)$ , we use the formulas in Lemma 1.4, by applying two surjective homomorphisms in Section 3

$$(4.5) \quad p_{k-1,1} : V(\varpi_{k-1})_{(-q^t)^{-1/t}} \otimes V(\varpi_1)_{(-q^t)^{k-1/t}} \twoheadrightarrow V(\varpi_k),$$

$$(4.6) \quad p_{k-1,1}^* : V(\varpi_k)_{(-q^t)^{-1/t}} \otimes V(\varpi_1)_{(p^*)(-q^t)^{-k/t}} \rightarrow V(\varpi_{k-1}),$$

and setting  $W = V(\varpi_l)$  or  $V(\varpi_n)$ , to get two elements in  $\mathbf{k}[z^{\pm 1}]$  which are described in terms of  $d_{k,l}(z)$ 's and  $a_{k,l}(z)$ 's. Here (4.6) is the composition of  $U'_q(\mathfrak{g})$ -homomorphisms given as follows:

$$\begin{aligned} V(\varpi_k)_{(-q^t)^{-1/t}} \otimes V(\varpi_1)_{(p^*)(-q^t)^{-k/t}} &\hookrightarrow V(\varpi_{k-1}) \otimes V(\varpi_1)_{(-q^t)^{-k/t}} \otimes V(\varpi_1)_{(p^*)(-q^t)^{-k/t}} \\ &\twoheadrightarrow V(\varpi_{k-1}) \otimes \mathbf{k} \simeq V(\varpi_{k-1}). \end{aligned}$$

Since we know the forms of  $a_{k,l}(z)$ 's, two elements in  $\mathbf{k}[z^{\pm 1}]$  can be described in terms of  $d_{k,l}(z)$ 's and polynomials in  $\mathbf{k}[z]$  (up to constant multiple of  $\mathbf{k}[z^{\pm 1}]^\times$ ).

By the assumptions, we know  $d_{1,1}(z)$ ,  $d_{1,n}(z)$  and hence we can compute  $d_{k,l}(z)$  and  $d_{k,n}(z)$ , by manipulating the two elements in  $\mathbf{k}[z^{\pm 1}]$  and using inductions.

The denominator  $d_{1,1}(z)$  of  $R_{1,1}^{\text{norm}}(z) : V(\varpi_1) \otimes V(\varpi_1)_z \rightarrow V(\varpi_1)_z \otimes V(\varpi_1)$  are computed in [17] (see also [13] for  $\mathfrak{g} = A_2^{(2)}$ ) as follows:

$$(4.7) \quad d_{1,1}(z) = (z^t - (q^2)^t)(z^t - (p^*)^t).$$

The denominator  $d_{1,n}(z)$  of  $R_{1,n}^{\text{norm}}(z) : V(\varpi_1) \otimes V(\varpi_n)_z \rightarrow V(\varpi_n)_z \otimes V(\varpi_1)$  for  $\mathfrak{g} = B_n^{(1)}$  is computed in [7] as follows:

$$(4.8) \quad d_{1,n}(z) = d_{n,1}(z) = z - (-1)^{n+1} q_s^{2n+1}.$$

Considering Assumption (4.1), the only missing part is the denominator  $d_{1,n}(z)$  for  $\mathfrak{g} = D_{n+1}^{(2)}$ .

**4.2. The denominator  $d_{1,n}(z)$  for  $\mathfrak{g} = D_{n+1}^{(2)}$ .** To compute the denominator  $d_{1,n}(z)$  for  $\mathfrak{g} = D_{n+1}^{(2)}$ , we follow the notations and arguments given in [17, Section 4].

By the  $U'_q(D_{n+1}^{(2)})$ -module structure of  $V(\varpi_1)$  and  $V(\varpi_n)$  in Section 2, we have

$$V(\varpi_1) \simeq V_0(\varpi_1) \oplus V_0(0) \text{ and } V(\varpi_n) \simeq V_0(\varpi_n) \text{ as } U_q(B_n)\text{-modules.}$$

Here  $V_0(\varpi_n)$  (resp.  $V_0(0)$ ) is the highest  $U_q(B_n)$ -module with the highest weight  $\varpi_n$  (resp. 0). Thus we have

$$V(\varpi_n) \otimes V(\varpi_1) \simeq V_0(\lambda) \oplus V_0(\varpi_n)^{\oplus 2} \text{ as a } U_q(B_n)\text{-module,}$$

where  $\lambda = (\frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ . Let

$$m_n^+ = (+, \dots, +) \quad \text{and} \quad m^i = (+, \dots, +, \frac{i}{-}, +, \dots, +) \quad (1 \leq i \leq n)$$

be the elements in  $V(\varpi_n)$ . Then we have the following lemmas by the direct calculation:

**Lemma 4.2.** *Let  $u_\lambda$ ,  $u_{\varpi_n}^1$  and  $u_{\varpi_n}^2$  be the  $U_q(B_n)$ -highest weight vectors with the weight  $\lambda$ ,  $\varpi_n$  and  $\varpi_n$  in  $V(\varpi_n)_x \otimes V(\varpi_1)_y$  respectively. Then we have*

- (a)  $u_\lambda = (m_n^+) \otimes v_1$ ,
- (b)  $u_{\varpi_n}^1 = [2]_0^{-1} (m_n^+) \otimes v_0$ ,
- (c)  $u_{\varpi_n}^2 = \sum_{k=1}^n (-1)^k q^{2k} (m^{n+1-k}) \otimes v_{n+1-k} + [2]_n^{-1} (m_n^+) \otimes v_0$ .

**Lemma 4.3.** *Let  $\tilde{u}_\lambda$ ,  $\tilde{u}_{\varpi_n}^1$  and  $\tilde{u}_{\varpi_n}^2$  be the  $U_q(B_n)$ -highest weight vectors with the weight  $\lambda$ ,  $\varpi_n$  and  $\varpi_n$  in  $V(\varpi_1)_y \otimes V(\varpi_n)_x$ , respectively. Then we have*

- (a)  $\tilde{u}_\lambda = (1) \otimes (m_n^+)$ ,
- (b)  $\tilde{u}_{\varpi_n}^1 = [2]_0^{-1} v_\emptyset \otimes (m_n^+)$ ,
- (c)  $\tilde{u}_{\varpi_n}^2 = \sum_{k=1}^n (-1)^{n+1-k} q^{-2(n+1-k)} v_k \otimes (m^k) + q^{-1} [2]_n^{-1} v_0 \otimes (m_n^+)$ .

Hence  $R_{1,n}^{\text{norm}} : V(\varpi_1)_y \otimes V(\varpi_n)_x \rightarrow V(\varpi_n)_x \otimes V(\varpi_1)_y$  can be expressed by

$$R_{1,n}^{\text{norm}}(\tilde{u}_\lambda) = u_\lambda \quad \text{and} \quad R_{1,n}^{\text{norm}}(\tilde{u}_{\varpi_n}^i) = \sum_{j=1}^2 a_{ji}^{\varpi_n} u_{\varpi_n}^j.$$

The following lemmas can be obtained by direct calculations.

**Lemma 4.4.** *For the highest weight vectors defined in Lemma 4.2, we have*

- (a)  $f_0(u_{\varpi_n}^1) = x^{-1} y^{-1} (q^{-1} x) u_\lambda$ ,
- (b)  $f_0(u_{\varpi_n}^2) = x^{-1} y^{-1} ((-1)^n q^{2n} y) u_\lambda$ ,
- (c)  $e_1 \cdots e_{n-1} e_n^{(2)} e_{n-1} \cdots e_2 e_1 e_0(u_{\varpi_n}^1) = (y) u_\lambda$ ,
- (d)  $e_1 \cdots e_{n-1} e_n^{(2)} e_{n-1} \cdots e_2 e_1 e_0(u_{\varpi_n}^2) = (q^{-1} x) u_\lambda$ ,

in  $V(\varpi_n)_x \otimes V(\varpi_1)_y$ .

**Lemma 4.5.** *For the highest weight vectors defined in Lemma 4.3, we have*

- (a)  $f_0(\tilde{u}_{\varpi_n}^1) = x^{-1} y^{-1} (x) \tilde{u}_\lambda$ ,
- (b)  $f_0(\tilde{u}_{\varpi_n}^2) = x^{-1} y^{-1} ((-1)^n q^{-2n-2} y) \tilde{u}_\lambda$ ,
- (c)  $e_1 \cdots e_{n-1} e_n^{(2)} e_{n-1} \cdots e_2 e_1 e_0(\tilde{u}_{\varpi_n}^1) = (q^{-1} y) \tilde{u}_\lambda$ ,
- (d)  $e_1 \cdots e_{n-1} e_n^{(2)} e_{n-1} \cdots e_2 e_1 e_0(\tilde{u}_{\varpi_n}^2) = (q^{-1} x) \tilde{u}_\lambda$ ,

in  $V(\varpi_1)_y \otimes V(\varpi_n)_x$ .

From these lemmas, we obtain

$$\begin{pmatrix} q^{-1} y^{-1} & (-1)^n q^{2n} x^{-1} \\ y & q^{-1} x \end{pmatrix} (a_{ij}^{\varpi_n}) = \begin{pmatrix} y^{-1} & (-1)^n q^{-2n-2} x^{-1} \\ q^{-1} y & q^{-1} x \end{pmatrix},$$

and hence

$$(a_{ij}^{\varpi_n}) = \frac{1}{z^2 + (-q^2)^{n+1}} \begin{pmatrix} qz^2 - (-1)^n q^{2n+1} & (-1)^n (q^{-2n-1} - q^{2n+1})z \\ (1 - q^2)z & z^2 - (-1)^n q^{-2n} \end{pmatrix},$$

where  $z = xy^{-1}$ .

Hence we can conclude that

$$(4.9) \quad d_{1,n}(z) = d_{n,1}(z) = z^2 + (-q^2)^{n+1} \quad \text{for } \mathfrak{g} = D_{n+1}^{(2)}.$$

**4.3. Denominators between fundamental representations.** Write

$$d_{k,l}(z) = \prod_{\nu} (z - x_{\nu}).$$

For rational functions  $f, g \in \mathbf{k}(z)$ , we write  $f \equiv g$  if there exists an element  $a \in \mathbf{k}[z^{\pm 1}]^{\times}$  such that

$$f = ag.$$

**Lemma 4.6.** [1] *For  $k, l \in I_0$ , we have*

$$(4.10) \quad \begin{aligned} a_{k,l}(z) a_{k,l}((p^*)^{-1} z) &\equiv \frac{d_{k,l}(z)}{d_{k,l}(p^* z^{-1})}, \\ a_{k,l}(z) &= q^{(\varpi_k, \varpi_l)} \prod_{\nu} \frac{(p^* x_{\nu} z; p^{*2})_{\infty} (p^* x_{\nu}^{-1} z; p^{*2})_{\infty}}{(x_{\nu} z; p^{*2})_{\infty} (p^{*2} x_{\nu}^{-1} z; p^{*2})_{\infty}}, \end{aligned}$$

where  $(z; q)_\infty = \prod_{s=0}^{\infty} (1 - q^s z)$ .

Now we list a table for triple  $(\delta, c, p^*)$  for each  $\mathfrak{g}$ :

$\mathfrak{g}$	$\delta$	$c$	$p^*$
$A_{2n-1}^{(2)}$	$\alpha_0 + \alpha_1 + 2(\alpha_2 + \cdots + \alpha_{n-1}) + \alpha_n$	$h_0 + h_1 + 2(h_2 + \cdots + h_n)$	$-(-q)^{2n}$
$A_{2n}^{(2)}$	$2(\alpha_0 + \cdots + \alpha_{n-1}) + \alpha_n$	$h_0 + 2(h_1 + \cdots + h_n)$	$(-q)^{2n+1}$
$B_n^{(1)}$	$\alpha_0 + \alpha_1 + 2(\alpha_2 + \cdots + \alpha_n)$	$h_0 + h_1 + 2(h_2 + \cdots + h_{n-1}) + h_n$	$-(-q)^{2n-1}$
$D_{n+1}^{(2)}$	$\alpha_0 + \alpha_1 + \cdots + \alpha_n$	$h_0 + 2(h_1 + \cdots + h_{n-1}) + h_n$	$-(-q^2)^n$

TABLE 1.  $(\delta, c, p^*)$  for each affine type

By Lemma 4.6 and (4.7), we can compute  $a_{1,1}(z)$  for all  $\mathfrak{g}$  as follows:

$$(4.11) \quad a_{1,1}(z) = \begin{cases} q \frac{\langle 2n+2 \rangle \langle 2n-2 \rangle}{\langle 2n \rangle^2} \frac{[4n][0]}{[2][4n-2]} & \text{if } \mathfrak{g} = A_{2n-1}^{(2)}, \\ q \frac{[2n+3][2n-1]}{[2n+1]^2} \frac{[4n+2][0]}{[2][4n]} & \text{if } \mathfrak{g} = A_{2n}^{(2)}, \\ q \frac{[2n+1][2n-3]}{[2n-1]^2} \frac{[4n-2][0]}{[2][4n-4]} & \text{if } \mathfrak{g} = B_n^{(1)}, \\ q \frac{\{n+1\}\{n-1\}}{\{n\}^2} \frac{\{2n\}\{0\}}{\{1\}\{2n-1\}} & \text{if } \mathfrak{g} = D_{n+1}^{(2)}, \end{cases}$$

where, for  $a \in \mathbb{Z}$  and  $b \in \frac{1}{2}\mathbb{Z}$ ,

$$[a] = ((-q)^a z; p^{*2})_\infty, \quad \langle a \rangle = (-(-q)^a z; p^{*2})_\infty \quad \text{and} \quad \{b\} = ((-q^2)^b z; p^{*2})_\infty \times (-(-q^2)^b z; p^{*2})_\infty.$$

Note that, for  $a \in \mathbb{Z}$  and  $b \in \frac{1}{2}\mathbb{Z}$ , we have

$$(4.12) \quad \begin{aligned} [a]/[a+4n] &\equiv z - (-q)^{-a} \text{ and } \langle a \rangle / \langle a+4n \rangle \equiv z + (-q)^{-a} & \text{if } \mathfrak{g} = A_{2n-1}^{(2)}, \\ [a]/[a+4n+2] &\equiv z - (-q)^{-a} & \text{if } \mathfrak{g} = A_{2n}^{(2)}, \\ [a]/[a+4n-2] &\equiv z - (-q)^{-a} & \text{if } \mathfrak{g} = B_n^{(1)}, \\ \{b\}/\{b+2n\} &\equiv z^2 - (-q^2)^{-2b} & \text{if } \mathfrak{g} = D_{n+1}^{(2)}. \end{aligned}$$

Following [15], [7, (3.12)] and [14, (3.7)], we recall the image of  $v_k \otimes v_l$  ( $k \neq l \in I_0$ ) under the normalized  $R$ -matrix

$$R_{1,1}^{\text{norm}}(z): V(\varpi_1) \otimes V(\varpi_1)_z \rightarrow V(\varpi_1)_z \otimes V(\varpi_1),$$

which is given by

$$(4.13) \quad R_{1,1}^{\text{norm}}(z)(v_k \otimes v_l) = \frac{(1 - (q^2)^t) z^{t \times \delta(k \succ l)}}{z^t - (q^2)^t} (v_k \otimes v_l) + \frac{q^t (z^t - 1)}{z^t - (q^2)^t} (v_l \otimes v_k).$$

Here  $\succ$  is the linear order on the labeling set of the basis of  $V(\varpi_1)$  (see [12, Section 8]).

**Proposition 4.7.** *For  $1 \leq k, l \leq n - \vartheta$ , we have*

$$(4.14) \quad a_{k,l}(z) \equiv \begin{cases} \frac{[|k-l|][4n-|k-l|]}{[k+l][4n-k-l]} \frac{\langle 2n+k+l \rangle \langle 2n-k-l \rangle}{\langle 2n+|k-l| \rangle \langle 2n-|k-l| \rangle} & \text{if } \mathfrak{g} = A_{2n-1}^{(2)}, \\ \frac{[|k-l|][4n+2-|k-l|]}{[k+l][4n+2-k-l]} \frac{[2n+1+k+l][2n+1-k-l]}{[2n+1+|k-l|][2n+1-|k-l|]} & \text{if } \mathfrak{g} = A_{2n}^{(2)}, \\ \frac{[|k-l|][2n+k+l-1][2n-k-l-1][2n-|k-l|-1]}{[k+l][2n+k-l-1][2n-k+l-1][2n-k-l-2]} & \text{if } \mathfrak{g} = B_n^{(1)}, \\ \frac{\{|\frac{k-l}{2}|\}\{2n-|\frac{k-l}{2}|\}\{n+\frac{k+l}{2}\}\{n-\frac{k+l}{2}\}}{\{\frac{k+l}{2}\}\{2n-\frac{k+l}{2}\}\{n+|\frac{k-l}{2}|\}\{n-|\frac{k-l}{2}|\}} & \text{if } \mathfrak{g} = D_{n+1}^{(2)}. \end{cases}$$

*Proof.* We prove only for the case when  $\mathfrak{g}$  is of type  $A_{2n-1}^{(2)}$ . For the other  $\mathfrak{g}$ , one can apply the same argument to prove our assertion. We first consider when  $k = 1$ .

By (4.11), our assertion for  $k = l = 1$  holds. Applying the commutative diagram (4.3) for  $k = 1$ , we have

$$(4.15) \quad a_{1,l-1}(-q^{-1}z)a_{1,1}((-q)^{l-1}z)v_{[1,\dots,l-1]} \otimes w \mapsto a_{1,l}(z)v_{[1,\dots,l-1,l]} \otimes v_1,$$

where

$$w = R_{1,1}^{\text{norm}}((-q)^{l-1}z)(v_1 \otimes v_l) = \frac{q((-q)^{l-1}z-1)}{(-q)^{l-1}z-q^2}v_l \otimes v_1 + \frac{(1-q^2)}{(-q)^{l-1}z-q^2}v_1 \otimes v_l.$$

Since  $v_{[1,\dots,l-1]} \otimes v_1$  vanishes under the map  $p_{l-1,1}$ , (4.15) indicates that

$$\begin{aligned} a_{1,l}(z) &= a_{1,l-1}(-q^{-1}z) a_{1,1}((-q)^{l-1}z) \frac{q((-q)^{l-1}z-1)}{(-q)^{l-1}z-q^2} \\ &\equiv a_{1,l-1}(-q^{-1}z) a_{1,1}((-q)^{l-1}z) \frac{[l-1]}{[4n+l-1]} \frac{[4n+l-3]}{[l-3]}. \end{aligned}$$

Hence our assertion for  $k = 1$  follows from an induction on  $l$ :

$$(4.16) \quad a_{1,l}(z) = a_{l,1}(z) \equiv \frac{[l-1][4n-l+1]}{[l+1][4n-l-1]} \frac{\langle 2n-l-1 \rangle \langle 2n+l+1 \rangle}{\langle 2n+l-1 \rangle \langle 2n-l+1 \rangle}.$$

By (4.1), we now assume  $2 \leq l \leq k \leq n$ . By the direct calculation, one can show that

$$f_{l-1}f_{l-2} \cdots f_1(v_{[1,\dots,k]} \otimes v_1) = v_{[1,\dots,k]} \otimes v_l \quad \text{and} \quad f_{l-1}f_{l-2} \cdots f_1(v_1 \otimes v_{[1,\dots,k]}) = v_l \otimes v_{[1,\dots,k]}.$$

Since  $R_{k,1}^{\text{norm}}$  is a  $U'_q(\mathfrak{g})$ -homomorphism and sends  $v_{[1,\dots,k]} \otimes v_1$  to  $v_1 \otimes v_{[1,\dots,k]}$ , we have

$$R_{k,1}^{\text{norm}}(z)(v_{[1,\dots,k]} \otimes v_l) = v_l \otimes v_{[1,\dots,k]}.$$

Thus, the image in (4.4),

$$a_{k,l-1}(-q^{-1}z)a_{k,1}((-q)^{l-1}z)v_{[1,\dots,l-1]} \otimes w \mapsto a_{k,l}(z)v_{[1,\dots,l-1,l]} \otimes v_{[1,\dots,k]}$$

for  $w = R_{k,1}^{\text{norm}}((-q)^{l-1}z)(v_{[1,\dots,k]} \otimes v_l) = v_l \otimes v_{[1,\dots,k]}$ , implies that

$$(4.17) \quad a_{k,l}(z) = a_{k,l-1}(-q^{-1}z) a_{k,1}((-q)^{l-1}z) \quad (2 \leq l \leq k \leq n).$$

Hence one can obtain our assertion by applying an induction on  $l$ . □

**Theorem 4.8.** *For  $1 \leq k, l \leq n - \vartheta$ , we have*

$$(4.18) \quad d_{k,l}(z) = \prod_{s=1}^{\min(k,l)} (z^t - (-q^t)^{|k-l|+2s})(z^t - (p^*)^t(-q^t)^{2s-k-l}).$$

*Proof.* For  $1 \leq k, l \leq n - \vartheta$ , set

$$(4.19) \quad D_{k,l}(z) = \prod_{s=1}^{\min(k,l)} (z^t - (-q^t)^{|k-l|+2s})(z^t - (p^*)^t(-q^2)^{2s-k-l}).$$

Then we can observe that  $D_{k,l}(z)$  behaves similar to  $d_{k,l}(z)$ . Namely, (cf. (4.1), (4.7) and (4.10))

$$(4.20) \quad D_{1,1}(z) = d_{1,1}(z), \quad D_{k,l}(z) = D_{l,k}(z),$$

$$(4.21) \quad \frac{D_{k,l}(z)}{D_{k,l}(p^*z^{-1})} \equiv a_{k,l}(z)a_{k,l}((p^*)^{-1}z) \equiv \frac{d_{k,l}(z)}{d_{k,l}(p^*z^{-1})}.$$

By calculations, one can check that

$$(4.22) \quad D_{k,l}(z) = D_{k,l-1}((-q^t)^{-1/t}z)D_{k,1}((-q^t)^{l-1/2}z) \quad \text{for } 2 \leq k \leq n - \vartheta,$$

which is similar to (4.17), also.

Now we give a proof for  $\mathfrak{g} = D_{n+1}^{(2)}$ , since this case is most complicated. For the other  $\mathfrak{g}$ , one can apply the similar argument to prove.

We shall show that  $D_{k,l}(z) = d_{k,l}(z)$  indeed. Our assertion for  $k = l = 1$  is presented in (4.9). Assume that  $1 \leq k \leq n - 1$  and  $2 \leq l \leq n - 1$ .

From the a surjective homomorphism in Theorem 3.5

$$p_{l-1,1}: V(\varpi_{l-1})_{(-q^2)^{-\frac{1}{2}}} \otimes V(\varpi_1)_{(-q^2)^{\frac{l-1}{2}}} \twoheadrightarrow V(\varpi_l),$$

the first formula in Lemma 1.4 with setting  $W = V(\varpi_k)$  yields an element in  $\mathbf{k}[z^{\pm 1}]$  as follows:

$$(4.23) \quad \frac{d_{k,l-1}((-q^2)^{-\frac{1}{2}}z)d_{k,1}((-q^2)^{\frac{l-1}{2}}z)}{d_{k,l}(z)} \frac{a_{k,l}(z)}{a_{k,l-1}((-q^2)^{-\frac{1}{2}}z)a_{k,1}((-q^2)^{\frac{l-1}{2}}z)} \in \mathbf{k}[z^{\pm 1}],$$

for  $1 \leq k \leq n - 1$  and  $2 \leq l \leq n - 1$ . In particular, if  $2 \leq l \leq k \leq n - 1$ ,

$$(4.24) \quad \frac{d_{k,l-1}((-q^2)^{-\frac{1}{2}}z)d_{k,1}((-q^2)^{\frac{l-1}{2}}z)}{d_{k,l}(z)} \in \mathbf{k}[z^{\pm 1}],$$

since (cf. (4.17))

$$\frac{a_{k,l}(z)}{a_{k,l-1}((-q^2)^{-\frac{1}{2}}z)a_{k,1}((-q^2)^{\frac{l-1}{2}}z)} \in \mathbf{k}[z^{\pm 1}]^\times$$

by the computation using (4.14).

Using (4.14) once again, for  $k = 1 < l$ , one can compute that

$$\frac{a_{1,l}(z)}{a_{1,l-1}((-q^2)^{-\frac{1}{2}}z)a_{1,1}((-q^2)^{\frac{l-1}{2}}z)} \equiv \frac{(z^2 - (-q^2)^{1-l})}{(z^2 - (-q^2)^{3-l})} \quad \text{for } 2 \leq l \leq n - 1.$$

Set  $k = 1$  and then replace  $l$  with  $k$  in (4.23). Then (4.23) becomes

$$(4.25) \quad \begin{aligned} & \frac{d_{1,k-1}((-q^2)^{-\frac{1}{2}}z)D_{1,1}((-q^2)^{\frac{k-1}{2}}z)}{d_{1,k}(z)} \frac{(z^2 - (-q^2)^{1-k})}{(z^2 - (-q^2)^{3-k})} \\ & \equiv \frac{d_{1,k-1}((-q^2)^{-\frac{1}{2}}z)(z^2 - (-q^2)^{2n-k+1})(z^2 - (-q^2)^{1-k})}{d_{1,k}(z)} \in \mathbf{k}[z^{\pm 1}] \quad \text{for } 2 \leq k \leq n - 1, \end{aligned}$$

since  $D_{1,1}(z) = d_{1,1}(z)$ .

On the other hand, from the surjective homomorphism

$$V(\varpi_k)_{(-q^2)^{-\frac{1}{2}}} \otimes V(\varpi_1)_{(-q^2)^{\frac{2n-k}{2}}} \rightarrow V(\varpi_{k-1}),$$

the second formula in Lemma 1.4 with setting  $W = V(\varpi_l)$  yields an element in  $\mathbf{k}[z^{\pm 1}]$  as follows:

$$(4.26) \quad \frac{d_{1,l}(-(-q^2)^{\frac{k-2n}{2}}z)d_{k,l}((-q^2)^{\frac{1}{2}}z)}{d_{k-1,l}(z)} \frac{a_{k-1,l}(z)}{a_{k,l}((-q^2)^{\frac{1}{2}}z)a_{1,l}(-(-q^2)^{\frac{k-2n}{2}}z)} \in \mathbf{k}[z^{\pm 1}].$$

By computation using (4.14), we have

$$\frac{a_{k-1,l}(z)}{a_{k,l}((-q^2)^{\frac{1}{2}}z)a_{1,l}(-(-q^2)^{\frac{k-2n}{2}}z)} \equiv \begin{cases} \frac{z^2 - (-q^2)^{2n-k-l-1}}{z^2 - (-q^2)^{2n-k-l+1}} & \text{if } 1 \leq l < k \leq n-1, \\ \frac{(z^2 - (-q^2)^{2n-k-l-1})(z^2 - (-q^2)^{-1})}{(z^2 - (-q^2)^{2n-k-l+1})(z^2 - (-q^2)^1)} & \text{if } 2 \leq l = k \leq n-1. \end{cases}$$

Thus the element (4.26) in  $\mathbf{k}[z^{\pm 1}]$  can be written as follows:

$$(4.27) \quad \frac{d_{1,l}(-(-q^2)^{\frac{k-2n}{2}}z)d_{k,l}((-q^2)^{\frac{1}{2}}z)}{d_{k-1,l}(z)} \frac{(z^2 - (-q^2)^{2n-k-l-1})}{(z^2 - (-q^2)^{2n-k-l+1})} \in \mathbf{k}[z^{\pm 1}] \quad \text{if } 1 \leq l < k \leq n-1,$$

and

$$(4.28) \quad \frac{d_{1,l}(-(-q^2)^{\frac{k-2n}{2}}z)d_{k,l}((-q^2)^{\frac{1}{2}}z)}{d_{k-1,l}(z)} \frac{(z^2 - (-q^2)^{2n-k-l-1})}{(z^2 - (-q^2)^{2n-k-l+1})} \frac{(z^2 - (-q^2)^{-1})}{(z^2 - (-q^2)^1)} \in \mathbf{k}[z^{\pm 1}]$$

if  $2 \leq l = k \leq n-1$ .

Setting  $l = 1$  in (4.27), we obtain

$$(4.29) \quad \frac{D_{1,1}(-(-q^2)^{\frac{k-2n-1}{2}}z)d_{k,1}(z)}{d_{k-1,1}((-q^2)^{-\frac{1}{2}}z)} \frac{(z^2 - (-q^2)^{2n-k-1})}{(z^2 - (-q^2)^{2n-k+1})} \in \mathbf{k}[z^{\pm 1}] \quad \text{for } 2 \leq k \leq n-1.$$

Now we claim that

$$d_{1,k}(z) = D_{1,k}(z) = (z^2 - (-q^2)^{k+1})(z^2 - (-q^2)^{2n-k+1}) \quad \text{for } 2 \leq k \leq n-1.$$

With (4.20), we can start an induction on  $k$ . Thus (4.25) can be written in the following form:

$$(4.30) \quad \frac{D_{1,k-1}((-q^2)^{-\frac{1}{2}}z)(z^2 - (-q^2)^{2n+1-k})(z^2 - (-q^2)^{1-k})}{d_{1,k}(z)} \quad \text{for } 2 \leq k \leq n-1 \\ \equiv \frac{(z^2 - (-q^2)^{k+1})(z^2 - (-q^2)^{2n-k+3})(z^2 - (-q^2)^{2n+1-k})(z^2 - (-q^2)^{1-k})}{d_{1,k}(z)} \in \mathbf{k}[z^{\pm 1}].$$

Now we claim that

$$(4.31) \quad z = \pm(-q^2)^{\frac{1-k}{2}}, \pm(-q^2)^{\frac{2n-k+3}{2}} \text{ are not zero of } d_{1,k}(z).$$

If (4.31) is true, we have

$$(4.32) \quad \frac{D_{1,k}(z)}{d_{1,k}(z)} = \frac{(z^2 - (-q^2)^{k+1})(z^2 - (-q^2)^{2n+1-k})}{d_{1,k}(z)} \in \mathbf{k}[z^{\pm 1}] \quad (2 \leq k \leq n-1).$$

Since  $\frac{1-k}{2} \leq 0$ ,  $\pm(-q^2)^{\frac{1-k}{2}} \notin \mathbb{C}[[q]]$ . Then [19, Theorem 2.2.1 (i)] tells that  $\pm(-q^2)^{\frac{1-k}{2}}$  can not be a zero of  $d_{1,k}(z)$ .

If  $z = \pm(-q^2)^{\frac{k-3}{2}}$  is a zero of  $d_{1,k}(z)$ , we have a contradiction to the fact that the element (4.32) is in  $\mathbf{k}[z^{\pm 1}]$ . Thus we know that  $z = \pm(-q^2)^{\frac{k-3}{2}}$  is not a zero of  $d_{1,k}(z)$ . Since  $\frac{D_{k,l}(z)}{d_{k,l}(z)} \equiv \frac{D_{k,l}(p^*z^{-1})}{d_{k,l}(p^*z^{-1})}$  by (4.21), one can check that

- $z = \pm(-q^2)^{\frac{k-3}{2}}$  is not a pole of  $D_{1,k}(z)/d_{1,k}(z)$ ,
- $z = \pm(-q^2)^{\frac{2n-k+3}{2}}$  is not a pole of  $D_{1,k}(-(-q^2)^nz^{-1})/d_{1,k}(-(-q^2)^nz^{-1})$ .

Thus  $\pm(-q^2)^{\frac{2n-k+3}{2}}$  can not be zero of  $d_{1,k}(z)$  and hence the claim in (4.31) holds.

By an induction on  $k$  in (4.29), we also obtain

$$\begin{aligned} \frac{d_{1,k}(z)}{(z^2 - (-q^2)^{k+1})(z^2 - (-q^2)^{2n+1-k})} &\in \mathbf{k}[z^{\pm 1}] && \text{if } k \neq n-1, \\ \frac{d_{1,k}(z)}{(z^2 - (-q^2)^{2n+1-k})} &\in \mathbf{k}[z^{\pm 1}] && \text{if } k = n-1. \end{aligned}$$

By Theorem 3.5 and Lemma 3.10,  $d_{1,k}(z)$  has zeros at  $\pm(-q^2)^{\frac{k+1}{2}}$  for  $1 \leq k \leq n-1$ . Thus we have

$$(4.33) \quad \frac{d_{1,k}(z)}{D_{1,k}(z)} = \frac{d_{1,k}(z)}{(z^2 - (-q^2)^{k+1})(z^2 - (-q^2)^{2n+1-k})} \in \mathbf{k}[z^{\pm 1}] \quad (2 \leq k \leq n-1).$$

By considering (4.32) and (4.33) together, our assertion for  $k=1$  holds:

$$d_{1,k}(z) = (z^2 - (-q^2)^{k+1})(z^2 - (-q^2)^{2n+1-k}) = D_{1,k}(z) \quad (2 \leq k \leq n-1).$$

Now we apply an induction on  $k+l$ . Applying the induction at (4.24) with (4.22), we have

$$\frac{d_{k,l-1}((-q^2)^{-\frac{1}{2}}z)d_{k,1}((-q^2)^{\frac{l-1}{2}}z)}{d_{k,l}(z)} = \frac{D_{k,l-1}((-q^2)^{-\frac{1}{2}}z)D_{k,1}((-q^2)^{\frac{l-1}{2}}z)}{d_{k,l}(z)} = \frac{D_{k,l}(z)}{d_{k,l}(z)} \in \mathbf{k}[z^{\pm 1}]$$

for  $2 \leq l \leq k \leq n-1$ .

Let  $\phi_{k,l}(z)$  be the elements in  $\mathbf{k}[z^{\pm 1}]$  satisfying  $D_{k,l}(z) = d_{k,l}(z)\phi_{k,l}(z)$ . We claim that

$$\phi_{k,l}(z) = 1 \quad \text{for } 2 \leq l \leq k \leq n-1.$$

Note that

$$\begin{aligned} &\frac{D_{1,l}(-(-q^2)^{\frac{k-2n}{2}}z)D_{k,l}((-q^2)^{\frac{1}{2}}z)}{D_{k-1,l}(z)} \frac{(z^2 - (-q^2)^{2n-k-l-1})}{(z^2 - (-q^2)^{2n-k-l+1})} \\ &= \begin{cases} (z^2 - (-q^2)^{4n-k-l+1})(z^2 - (-q^2)^{2n-k-l-1}) & \text{if } l < k, \\ (z^2 - (-q^2)^{4n-k-l+1})(z^2 - (-q^2)^{2n-k-l-1})(z^2 - (-q^2)^{2n+1})(z^2 - (-q^2)) & \text{if } l = k. \end{cases} \end{aligned}$$

By (4.27), (4.28) and an induction on  $k+l$ , the above elements are written in the following form:

$$\begin{aligned} &\frac{(z^2 - (-q^2)^{4n-k-l+1})(z^2 - (-q^2)^{2n-k-l-1})}{\phi_{k,l}((-q^2)^{\frac{1}{2}}z)} \in \mathbf{k}[z^{\pm 1}] \quad \text{if } l < k, \\ &\frac{(z^2 - (-q^2)^{4n-k-l+1})(z^2 - (-q^2)^{2n-k-l-1})(z^2 - (-q^2)^{2n+1})(z^2 - (-q^2)^{-1})}{\phi_{k,l}((-q^2)^{\frac{1}{2}}z)} \in \mathbf{k}[z^{\pm 1}] \quad \text{if } l = k. \end{aligned}$$

Since  $\phi_{k,l}((-q^2)^{\frac{1}{2}}z)$  divides  $D_{k,l}((-q^2)^{\frac{1}{2}}z)$ , we conclude that

$$(4.34) \quad \begin{aligned} &\phi_{k,l}(z) = 1 && \text{if } k+l < n, \\ &\frac{(z^2 - (-q^2)^{2n-k-l})}{\phi_{k,l}(z)} \in \mathbf{k}[z^{\pm 1}] && \text{if } k+l \geq n. \end{aligned}$$

Now our assertion holds if  $z = \pm(-q^2)^{\frac{2n-k-l}{2}}$  is not a zero of  $\phi_{k,l}(z)$  for  $k+l \geq n$ . From (4.21), one can see that  $\phi_{k,l}(-(-q^2)^n z^{-1}) \equiv \phi_{k,l}(z)$ . Thus we suffice to prove that  $z = \pm(-q^2)^{\frac{k+l}{2}}$  is not a zero of  $\phi_{k,l}(z)$  for  $k+l \geq n$ . If  $k+l > n$ , then we have  $n > 2n-k-l$  and hence  $\phi_{k,l}(z) = 1$ .

Now we consider when  $k+l = n$ . Then Lemma 3.10 tells that  $d_{k,l}(z)$  has zeros at  $z = \pm(-q^2)^{\frac{k+l}{2}}$ . By the definition of  $D_{k,l}(z)$ ,  $\pm(-q^2)^{\frac{k+l}{2}}$  is a zero of multiplicity 1. Thus  $\pm(-q^2)^{\frac{k+l}{2}}$  can not be a zero of  $\phi_{k,l}(z)$  when  $k+l = n$ .  $\square$

Now we shall compute  $d_{k,n}(z)$  for  $\mathfrak{g} = B_n^{(1)}$  and  $\mathfrak{g} = D_{n+1}^{(2)}$ . By Lemma 4.6, (4.8) and (4.9), we have

$$(4.35) \quad a_{1,n}(z) \equiv \begin{cases} \frac{[2n-3]_{(n+1)}[6n-1]_{(n+1)}}{[2n+1]_{(n+1)}[6n-5]_{(n+1)}} & \text{if } \mathfrak{g} = B_n^{(1)}, \\ \frac{\{\frac{3n+1}{2}\}'\{\frac{n-1}{2}\}'}{\{\frac{3n-1}{2}\}'\{\frac{n+1}{2}\}'} & \text{if } \mathfrak{g} = D_{n+1}^{(2)}, \end{cases}$$

where, for  $a, k \in \mathbb{Z}$  and  $b \in \frac{1}{2}\mathbb{Z}$ ,

$$[a]_{(k)} = ((-1)^k q_s^a z; p^{*2})_\infty \quad \text{and} \quad \{b\}' = (-\sqrt{-1}(-q^2)^b; p^{*2})_\infty (\sqrt{-1}(-q^2)^b; p^{*2})_\infty.$$

Now, we give a proof only for  $\mathfrak{g} = B_1^{(n)}$ . For  $\mathfrak{g} = D_{n+1}^{(2)}$ , one can apply the same arguments.

**Proposition 4.9.** *For  $1 \leq l \leq n-1$ , we have*

$$(4.36) \quad a_{l,n}(z) \equiv \begin{cases} \frac{[2n-2l-1]_{(n+l)}[6n+2l-3]_{(n+l)}}{[2n+2l-1]_{(n+l)}[6n-2l-3]_{(n+l)}} & \text{if } \mathfrak{g} = B_n^{(1)}, \\ \frac{\{\frac{3n+l}{2}\}'\{\frac{n-l}{2}\}'}{\{\frac{3n-l}{2}\}'\{\frac{n+l}{2}\}'} & \text{if } \mathfrak{g} = D_{n+1}^{(2)}. \end{cases}$$

*Proof.* By (4.35), it suffices to consider when  $2 \leq l \leq n-1$ . Applying the commutative diagram (4.3) with setting  $k = n$ , (4.4) tells that we have

$$a_{n,l-1}(-q^{-1}z)a_{n,1}((-q)^{l-1}z)v_{[1,\dots,l-1]} \otimes w \mapsto a_{n,l}(z)v_{[1,\dots,l-1,l]} \otimes m_n^+,$$

where  $w = R_{n,1}^{\text{norm}}((-q)^{l-1}z)(m_n^+ \otimes v_l)$  for the highest weight vector  $m_n^+$  of  $V(\varpi_n)$ .

Since  $m_n^+$  vanishes by the action  $f_i$  ( $1 \leq i \leq l-1$ ), as in the proof of Proposition 4.7,

$$w = R_{n,1}^{\text{norm}}((-q)^{l-1}z)(m_n^+ \otimes v_l) = v_l \otimes m_n^+,$$

and hence

$$(4.37) \quad a_{n,l}(z) = a_{n,l-1}(-q^{-1}z) a_{n,1}((-q)^{l-1}z) \quad \text{for } 2 \leq l \leq n-1.$$

By (4.35) and an induction on  $l$ , our assertion follows.  $\square$

**Theorem 4.10.** *For  $1 \leq k \leq n-1$ , we have*

$$(4.38) \quad d_{k,n}(z) = \begin{cases} \prod_{s=1}^k (z - (-1)^{n+k} q_s^{2n-2k-1+4s}) & \text{if } \mathfrak{g} = B_n^{(1)}, \\ \prod_{s=1}^k (z^2 + (-q^2)^{n-k+2s}) & \text{if } \mathfrak{g} = D_{n+1}^{(2)}. \end{cases}$$



*Proof.* By (4.8), it suffices to consider when  $2 \leq k \leq n-1$ . From the surjective homomorphism in Theorem 3.5

$$V(\varpi_{k-1})_{(-q)^{-1}} \otimes V(\varpi_1)_{(-q)^{k-1}} \twoheadrightarrow V(\varpi_k),$$

the first formula in Lemma 1.4 with  $W = V(\varpi_n)$  yields an element in  $\mathbf{k}[z^{\pm 1}]$  as follows:

$$\frac{d_{k-1,n}(-q^{-1}z)d_{1,n}((-q)^{k-1}z)}{d_{k,n}(z)} \frac{a_{k,n}(z)}{a_{k-1,n}(-q^{-1}z)a_{1,n}((-q)^{k-1}z)} \in \mathbf{k}[z^{\pm 1}].$$

By (4.37), the element is written in more simplified form as follows:

$$(4.39) \quad \frac{d_{k-1,n}(-q^{-1}z)d_{1,n}((-q)^{k-1}z)}{d_{k,n}(z)} \equiv \frac{d_{k-1,n}(-q^{-1}z)(z - (-1)^{n+k}q_s^{2n-2k+3})}{d_{k,n}(z)} \in \mathbf{k}[z^{\pm 1}].$$

On the other hand, for each  $2 \leq k \leq n-1$ , we have a surjective homomorphism

$$V(\varpi_k)_{-q^{-1}} \otimes V(\varpi_1)_{(-q)^{2n-1-k}} \twoheadrightarrow V(\varpi_{k-1}).$$

Then the second formula in Lemma 1.4 with  $W = V(\varpi_n)$  yields an element in  $\mathbf{k}[z^{\pm 1}]$  as follows:

$$(4.40) \quad \frac{d_{1,n}(-(-q)^{k+1-2n}z)d_{k,n}(-qz)}{d_{k-1,n}(z)} \frac{a_{k-1,n}(z)}{a_{1,n}(-(-q)^{k+1-2n}z)a_{k,n}(-qz)} \in \mathbf{k}[z^{\pm 1}].$$

Using (4.36), the second factor of (4.40) can be written as

$$\frac{a_{k-1,n}(z)}{a_{1,n}(-(-q)^{k+1-2n}z)a_{k,n}(-qz)} \equiv \frac{z - (-1)^{n+k+1}q_s^{2n-2k-3}}{z - (-1)^{n+k+1}q_s^{2n-2k+1}}.$$

and hence (4.40) becomes

$$\frac{d_{k,n}(-qz)}{d_{k-1,n}(z)} \frac{(z - (-1)^{n+k+1}q_s^{6n-2k-1})(z - (-1)^{n+k+1}q_s^{2n-2k-3})}{(z - (-1)^{n+k+1}q_s^{2n-2k+1})} \in \mathbf{k}[z^{\pm 1}].$$

By the induction hypothesis,  $z = (-1)^{n+k+1}q_s^{6n-2k-1}$  and  $(-1)^{n+k+1}q_s^{2n-2k-3}$  are not zeros of  $d_{k-1,n}(z)$ . Hence we can conclude that

$$(4.41) \quad \frac{d_{k,n}(-qz)}{d_{k-1,n}(z)(z - (-1)^{n+k+1}q_s^{2n-2k+1})} \in \mathbf{k}[z^{\pm 1}],$$

which is equivalent to

$$\frac{d_{k,n}(z)}{d_{k-1,n}(-q^{-1}z)(z - (-1)^{n+k}q_s^{2n-2k+3})} \in \mathbf{k}[z^{\pm 1}].$$

Considering (4.39) and (4.41) together, our assertion follows:

$$d_{k,n}(z) \equiv d_{k-1,n}(-q^{-1}z)(z - (-1)^{n+k}q_s^{2n-2k+3}) = \prod_{s=1}^k (z - (-1)^{n+k}q_s^{2n-2k-1+4s}).$$

□

**Remark 4.11.** In conclusion, we can observe that

for all  $1 \leq k \leq n$ ,  $R_{k,l}^{\text{norm}}(z)$  has only simple poles unless  $\mathfrak{g} = D_{n+1}^{(2)}$ .

For  $\mathfrak{g} = D_{n+1}^{(2)}$ ,  $R_{k,l}^{\text{norm}}(z)$  has a double pole at  $z = \pm(-q^2)^{s/2}$  if

$$2 \leq k, l \leq n-1, \quad k+l > n, \quad 2n+2-k-l \leq s \leq k+l \quad \text{and} \quad s \equiv k+l \pmod{2}.$$

## APPENDIX A. THE TABLE OF DENOMINATORS.

Type	$n$	$k, l$	Denominators
$A_n^{(1)}$	$n \geq 1$	$1 \leq k, l \leq n$	$d_{k,l}(z) = \prod_{s=1}^{\min(k,l,n+1-k,n+1-l)} (z - (-q)^{2s+ k-l })$
$B_n^{(1)}$ $q_s^2 = q$	$n \geq 3$	$1 \leq k, l \leq n-1$	$d_{k,l}(z) = \prod_{s=1}^{\min(k,l)} (z - (-q)^{ k-l +2s})(z + (-q)^{2n-k-l-1+2s})$
		$1 \leq k \leq n-1$	$d_{k,n}(z) = \prod_{s=1}^k (z - (-1)^{n+k} q_s^{2n-2k-1+4s})$
		$k = l = n$	$d_{n,n}(z) = \prod_{s=1}^n (z - (q_s)^{4s-2})$
$C_n^{(1)}$	$n \geq 2$	$1 \leq k, l \leq n$	$d_{k,l}(z) = \prod_{s=1}^{\min(k,l,n-k,n-l)} (z - (-q_s)^{ k-l +2s}) \prod_{i=1}^{\min(k,l)} (z - (-q_s)^{2n+2-k-l+2s})$
$D_n^{(1)}$	$n \geq 4$	$1 \leq k, l \leq n-2$	$d_{k,l}(z) = \prod_{s=1}^{\min(k,l)} (z - (-q)^{ k-l +2s})(z - (-q)^{2n-2-k-l+2s})$
		$1 \leq k \leq n-2$	$d_{k,n-1}(z) = d_{k,n}(z) = \prod_{s=1}^k (z - (-q)^{n-k-1+2s})$
		$\{k, l\} = \{n, n-1\}$	$d_{n,n-1}(z) = d_{n-1,n}(z) = \prod_{s=1}^{\lfloor \frac{n-1}{2} \rfloor} (z - (-q)^{4s})$
		$k = l \in \{n, n-1\}$	$d_{n,n}(z) = d_{n-1,n-1}(z) = \prod_{s=1}^{\lfloor \frac{n}{2} \rfloor} (z - (-q)^{4s-2})$
$A_{2n-1}^{(2)}$	$n \geq 3$	$1 \leq k, l \leq n$	$d_{k,l}(z) = \prod_{s=1}^{\min(k,l)} (z - (-q)^{ k-l +2s})(z + (-q)^{2n-k-l+2s})$
$A_{2n}^{(2)}$	$n \geq 1$	$1 \leq k, l \leq n$	$d_{k,l}(z) = \prod_{s=1}^{\min(k,l)} (z - (-q)^{ k-l +2s})(z - (-q)^{2n+1-k-l+2s})$
$D_{n+1}^{(2)}$	$n \geq 2$	$1 \leq k, l \leq n-1$	$d_{k,l}(z) = \prod_{s=1}^{\min(k,l)} (z^2 - (-q^2)^{ k-l +2s})(z^2 - (-q^2)^{2n-k-l+2s})$
		$1 \leq k \leq n-1$	$d_{k,n}(z) = \prod_{s=1}^k (z^2 + (-q^2)^{n-k+2s})$
		$k = l = n$	$d_{n,n}(z) = \prod_{s=1}^n (z + (-q^2)^s)$

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